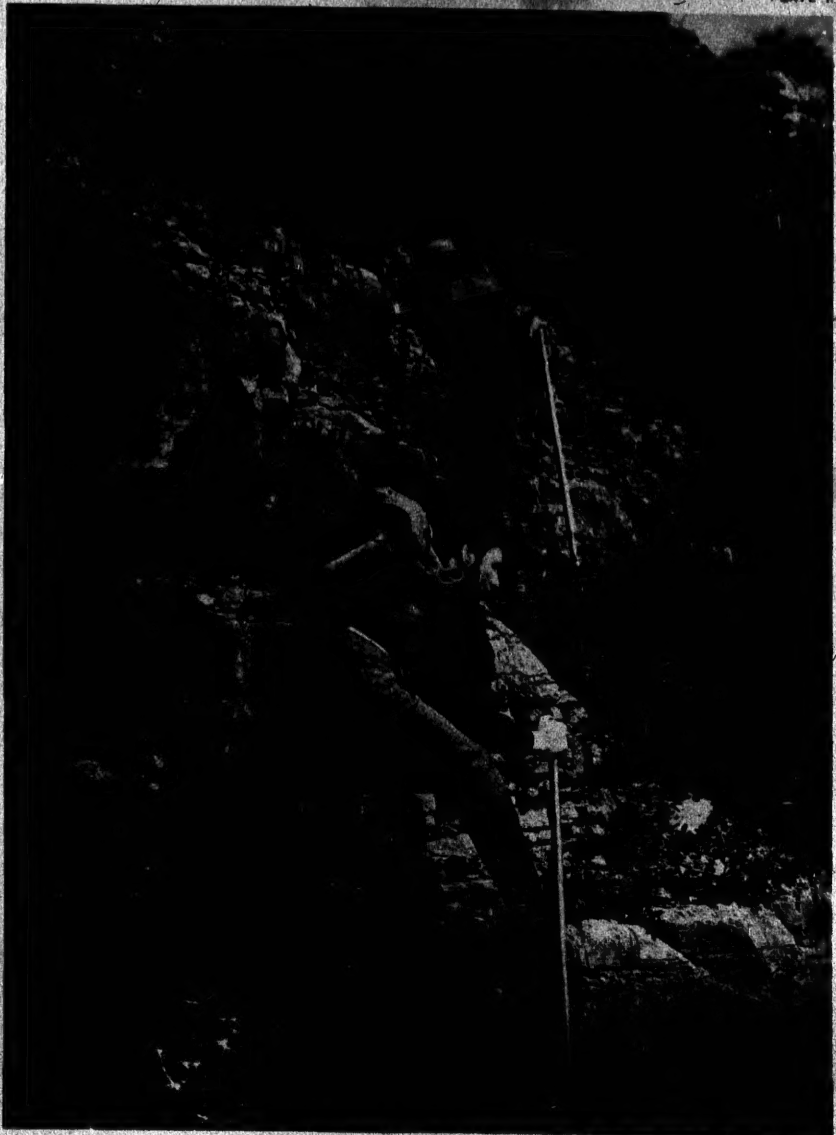
PHOTOGRAPHIC SURVEYING



TOPOGRAPHER AND ASSISTANT

SHOWING EQUIPMENT AND MODE OF CARRYING INSTRUMENTS.

PHOTOGRAPHIC SURVEYING

INCLUDING

THE ELEMENTS OF

DESCRIPTIVE GEOMETRY

AND

PERSPECTIVE

BY

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PREFACE

When the surveys of Dominion Lands were extended to the Rocky Mountains region, it was found that the methods hitherto employed were inadequate. The operations in the prairies consisted merely in defining the boundaries of the townships and sections: these lines form a network over the land by means of which the topographical features, always scarce in the prairies, are sufficiently well determined for general purposes.

In passing to the mountains, the conditions are entirely different; the topographical features are well marked and numerous, and the survey of the section lines is always difficult, often impossible and in most cases useless. The proper administration of the country required a tolerably accurate map: means had to be found of executing it rapidly and at a moderate cost.

The ordinary methods of topographical surveying were too slow and expensive for the purpose;rapid surveys based on a triangulation and on sketches were tried and proved ineffectual,then photography was resorted to and the results have been all that could be desired.

The application of photography to surveying is as old as the art itself. Arago,in presenting Daguerre's discovery,pointed out its application to surveying,but it was not until twenty-five years later that Laussedat gave in the "Mémorial de l'Officier du Génie" a full exposition of the method. His work was so complete that little has been added to it since.

In Germany,the principal exponent of the system has been an architect,Meydenbauer;his investigations were continued by Dörgens,Stolze,Vogel,Jordan and others.

In Italy,the celebrated Engineer Porro,to whom so many remarkable inventions are due,was the first to give his attention to the process:his ideas were followed by others and ultimately brought out the

Ordnance photographic surveys of the present day. To Major General Annibale Ferrero, present Director of the Geographical Military Institute, is due the credit of initiating these surveys: their execution was entrusted to Engineer L.Pio Paganini, with a staff of able assistants. The work of the Institute is very remarkable and deserves careful study.

Notwithstanding the number of those who have written on the method, the great advantages assigned to it and the numerous experimental surveys executed, there are but two examples of its actual employment for practical purposes: the Italian and the Canadian Surveys. In France, where it originated, it has been completely abandoned, at least ostensibly: the Germans use it for making plans of buildings, for which it is admirably adapted, but their topographical surveys have been merely experimental.

The failure is due to several causes. In the first place, the scales employed 1/5000 to 1/1000, are too large. On these scales, such objects as tables and chairs in a room could be shown on the plan, still no one would think of taking photographs of a room for making a plan of it. Smaller scales are not generally required in Europe, because good maps on these scales are already in existence.

On the other hand, the possibilities of the method have been over-estimated: it has been asserted that it would apply to almost any country; in reality there are but two classes of surveys to which it is well adapted, and they are the surveys in a mountainous country and the secret surveys. The Italian and Canadian Surveys are of the former description: the latter are in all probability extensively practised, although little is heard about them.

The authors who have written on the subject had in view surveys on a large scale, executed with great precision. The Canadian Surveys are quite different: they are on a small scale and the rapidity of execution is such that the more precise processes are not always available. The object of these notes is to show the amount of information which can be extracted from a photograph under various circumstances and the numerous processes at the disposal of the surveyor. The resources of photography in that respect, are unequalled by any other surveying method.

Considerable space has been devoted to perspective instruments; as now constructed, they are useless for purposes of topographical surveying, but there is no reason why they should not be made sufficiently precise, all that is required being more perfect workmanship. F[illegible] architectural surveys they may probably be employed with advantage.

In order to demonstrate more completely the method, a few explanations are given on secret and balloon surveying: although the subject is of no practical interest for Canadian surveyors, a knowledge of everything pertaining to photographic surveying cannot fail to prove useful to those engaged on photographic surveys.

A complete description of all the instruments invented and of the investigations of the various authors being given in the excellent work of Lieut. Henry A. Reed, U.S.A. "Photography applied to surveying", it has not been thought necessary to go again over the same ground.

The Canadian Photographic Surveys were commenced in 1887 and cover now over one thousand square miles: they are by far the most extensive ever made. For this reason, if for no other, it is hoped that the exposition of their mode of execution will prove acceptable to those interested in the development of the science of surveying.

While this was being printed three new books on the same subject were published in Italy, France and Germany: they are a great advance in the treatment of the subject and show that the method is slowly but surely gaining ground among practical men.

I have received much valuable assistance from Messrs O.J.Klotz, Dominion Topographical Surveyor and W.F.King, Chief Inspector of Surveys who kindly undertook to revise the proofs. I am indebted to them for many corrections and important changes in the text.

E.Deville.

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PHOTOGRAPHIC SURVEYING

CHAPTER I.

DESCRIPTIVE GEOMETRY

1. DEFINITIONS,PLANES OF PROJECTION. The object of descriptive geometry is to represent bodies and to solve problems on figures in space by means of their projections on certain planes called "Planes of projection"

2. GROUND LINE. For this purpose,two planes intersecting each other are employed:they divide space into four solid angles. Usually,one of the planes is vertical and the other one horizontal;their line of intersection is called "Ground line" and is denoted by the letters XY.

3. REPRESENTATION OF A POINT. Let XAY,,Fig.1,be the vertical plane,XBY the horizontal or ground

plane and P a point in space. From P, draw the perpendiculars Pp, Pp' to the ground and vertical planes; p is the horizontal projection of the point P and p' its vertical projection.

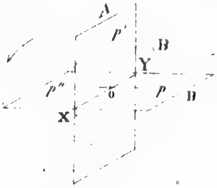

Fig.1

Let the vertical plane be revolved round the line XY as an axis, until it coincides with the ground plane; the point p' will fall at a point p'' such that the line pp'' will be perpendicular to XY.

For let a plane be drawn through Pp and Pp'; it is perpendicular to the ground plane as containing Pp and perpendicular to the vertical plane as containing Pp'; but when a plane is perpendicular to two other planes, it is perpendicular to their intersection, therefore the plane pPp' is perpendicular to XY, and its traces op and op' on the ground and vertical planes, are also perpendicular to XY, since a line perpendicular to a plane is perpendicular to all the lines passing through its foot in the plane.

But op' being perpendicular to XY, op'' must also be perpendicular to XY; it follows that pop'' is a straight line perpendicular to XY. The ground plane will then be as shown in Fig.2, op' being the distance of the point P from the ground plane and op its distance from the vertical plane; both points p and p' are on the same perpendicular to the ground line, as explained above.

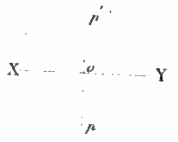

Fig.2

It is usual to represent points in space by capital letters, the horizontal projections by italic letters and the vertical projections by the same italic letters accented.

It has been shown that the two projections of a point are on a perpendicular to the ground line. Inversely, any two points on a perpendicular to the ground line are the projections of a point of space.

For let the part of Fig.2 above the ground line be revolved round the line XY until its plane be vertical as in Fig.1. Through p draw a parallel to op' and through p' a parallel to op: they will meet in a point P.

But op' is perpendicular to XY by hypothesis and it is also perpendicular to op, since pop' is the angle of the vertical and ground planes; therefore op' is perpendicular to the ground plane, because it is perpendicular to two lines in that plane. It follows that pP, parallel to op' is perpendicular to the same plane.

In the same manner it may be shown that p'P is perpendicular to the vertical plane:-

Therefore p and p' are the horizontal and vertical projections of the point P.

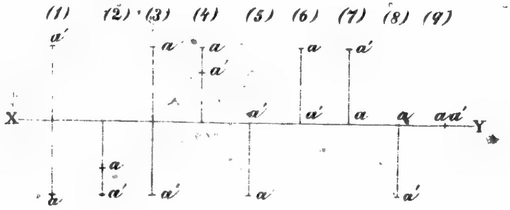

Fig.3

Fig.3 shows the representation of a point in various positions.

(1) is a point in front of the vertical plane and above the ground plane.

(2) is in front of the vertical plane and below the ground plane.

(3) is behind the vertical plane and below the ground plane.

(4) is behind the vertical plane and above the ground plane.

(5) is in the ground plane in front of the ground line.

(6) is also in the ground plane but behind the ground line.

(7) is in the vertical plane above the ground line.

(8) is also in the vertical plane but below the ground line.

(9) is on the ground line.

4. REPRESENTATION OF A STRAIGHT LINE. If perpendiculars be drawn to a plane from every point of a straight line, the locus of the feet of the perpendiculars is a straight line and is the orthogonal projection of the first one.

The projection of a straight line may also be defined as the intersection of one of the planes of projection by a second plane perpendicular to the first one and containing the given line. This second plane is called the projecting plane.

A straight line is perfectly defined by its projections, because it is the intersection of the two projecting planes. There is however an exception when the given line is contained in a plane perpendicular to the ground line; the two projecting planes coincide and the projections of the line are not sufficient to define it: the traces must be given.

The "traces" of a line are the points where it intersects the planes of projection. These points are easily found by noting that the vertical trace, (Fig.4) being in the vertical plane, its horizontal projection must be on the ground line, but it is also on the horizontal projection, ab , of the given line, therefore it must be at the intersection of the latter with the ground line at c. By erecting at c a perpendicular to the ground line, the vertical trace will be found at c'.

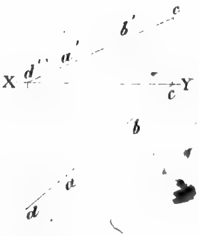

Fig.4

Similarly, the horizontal trace is obtained by erecting at d' a perpendicular $d'd$ to the ground line; d is the horizontal trace.

Inversely, the projections of a line may be obtained from the traces. By drawing a perpendicular from the vertical trace, *c'* Fig.4, to the ground line, a point *c* of the horizontal projection is obtained, which joined to the horizontal trace, *d*, gives the horizontal projection, *cd*. The vertical projection is obtained in a similar manner by finding the vertical projection *d'* of the horizontal trace *d*, and joining *c'd'*.

A straight line is defined by the projections of two of its points. Let *aa'*, *bb'*, Fig.4, be the points. The projections of the line will be *a'b'*, *ab*.

A straight line may occupy various positions with reference to the planes of projection; these positions are illustrated below.

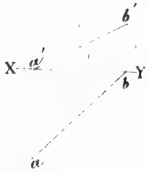

Fig.5

Fig.5 shows a line intersecting the vertical plane at *b'*, above the ground line and the ground plane at *a*, in front of the vertical plane.

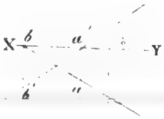

Fig.6

Fig.6 The vertical trace, *b'*, is below the ground line; the horizontal trace, *a*, is in front of it. The portion of the line between the traces is in the lower front dihedral angle.

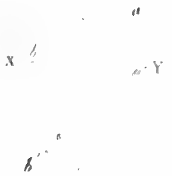

Fig.7

Fig.7 The vertical trace, b' is below the ground line;the horizontal trace, a, is behind it.

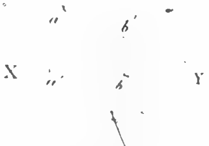

Fig.8

Fig.8. The vertical trace, b', is above the ground line;the horizontal trace, a is behind it.

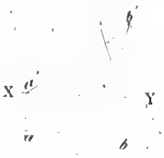

Fig.9

Fig.9. Line parallel to the vertical plane,with horizontal trace at a. In this case,the projecting plane through ab is parallel to the vertical plane, therefore its intersection with the ground plane, ab, is parallel to the ground line.

Fig.10

Fig 10. Horizontal line intersecting the vertical plane at a' The projecting plane through, $a'b'$ is parallel

to the ground plane, therefore its intersection with the vertical plane, *a'b'* is parallel to the ground line.

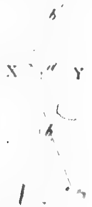

Fig. 11

Fig. 11. any line in a plane perpendicular to the ground line. The horizontal and vertical projections coincide and are on a perpendicular to the ground line. As explained above, the line in this case is not defined by its projections, which do not change, whatever may be the direction of the line in the projecting plane, but when the traces are given, the line is defined.

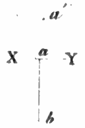

Fig. 12

Fig. 12. Line perpendicular to the vertical plane at *a'*. The vertical projection is a point, *a'*, since any perpendicular to the vertical plane from a point of the given line will intersect the plane at *a'*. The horizontal projection, *ab*, is a line perpendicular to the ground line, because the projecting plane is perpendicular to the two planes of projection and therefore is perpendicular to their intersection XY. There is no horizontal trace

b

Fig. 13. Line perpendicular to the ground plane at a. The perpendicular to the ground plane from any point of the given line will intersect the plane at a, which is the horizontal projection of the line. The vertical projection, $a'b'$, is perpendicular to the ground line, because the projecting plane, being perpendicular to both planes of projection, is perpendicular to their intersection, the ground line.

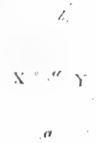

Fig. 13

Fig. 14. Line parallel to the ground line. In this case, each of the projecting planes is parallel to the ground line, therefore their intersections with the corresponding planes of projection are also parallel to the ground line.

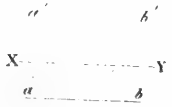

Fig. 14

Fig. 15. Line intersecting the ground line. The point of intersection, a, is at the same time the horizontal and the vertical trace of the line.

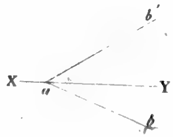

Fig. 15

and both projections intersect there.

When a line is in the ground plane, its horizontal projection is the line itself and its vertical projection is the ground line.

When a line is in the vertical plane, its vertical projection is the line itself and its horizontal projection is the ground line.

The ground line is its own horizontal projection and its own vertical projection.

5. TH[illegible]UGH A GIVEN POINT, DRAW A PARALLEL TO A GIVEN LINE. When two lines are parallel, their projections of same denomination are also parallel, because their projecting planes being perpendicular to the same plane of projection and passing through parallel lines, are themselves parallel to each other and therefore their intersections with the plane of projection are parallel lines.

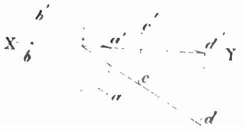

Fig.16

It follows that when a parallel to a line *ab, a'b'*, Fig.16 has to be drawn through a point *c,c'*, it is sufficient to draw through *c* a parallel to *ab* and through c' a parallel to $a'b'$ then $cd, c'd'$ is parallel to ab, $a'b'$.

When two lines $ab, a'b'; cd, c'd'$, (Fig.17) intersect each other, the points of intersection p and p', of the projections are on the same perpendicular to the ground line. It has been shown in § 3 that this is necessary in order that p and p' may represent a point in space.

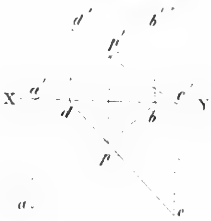

Fig.17

It follows that when the points p and p' are not on the same perpendicular to the ground line, the lines ab, $a'b'$; $cd, c'd'$, do not intersect, that is to say they are not contained in one plane.

6. REPRESENTATION OF A PLANE.

A plane is represented by its traces on the planes of projection, that is to say by its intersections with the said planes. These traces meet in a point α, Fig.18, of the ground line, which is the point where the plane cuts it. The vertical trace of the plane is $\alpha P'$, the horizontal trace is αP

When the plane is vertical, its trace $\alpha P'$, Fig. 19 on the vertical plane being the intersection of two vertical planes, is a vertical line.

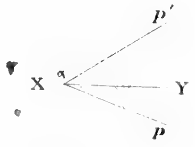

Fig. 18

But a vertical line is perpendicular to the ground plane and to all lines contained in this plane by which it is intersected: therefore it is perpendicular to the ground line.

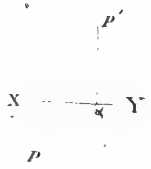

Fig. 19

It may be shown in the same way that the horizontal trace αP, Fig. 20, of a plane perpendicular to the vertical plane, is a line perpendicular to the ground line.

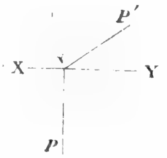

Fig. 20

The plane may be parallel to the vertical plane, in which case the vertical trace disappears. The horizontal trace, PQ, Fig. 21, is parallel to the ground line, because the two lines are the intersections of two parallel planes by a third one.

Fig. 21

Similarly, a horizontal plane is represented by its vertical trace, P'Q', Fig. 22, parallel to the ground line: the horizontal trace disappears.

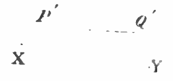

Fig. 22

When the plane is parallel to the ground line, the two traces PQ, P'Q', Fig. 23, are also parallel to it.

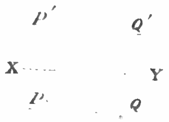

Fig. 23

A plane $P\alpha P'$, Fig. 24, perpendicular to the ground line, has its traces perpendicular to it. The ground line being

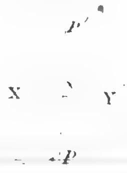

Fig. 24

perpendicular to the plane, is perpendicular to all the lines passing through α in that plane and therefore is perpendicular to the traces αP, $\alpha P'$.

Two parallel planes have their traces parallel, because the traces are then the intersections of two parallel planes by a third one.

7. LINE CONTAINED IN A PLANE. A line contained in a plane has its traces on the traces of the plane, since any point of the planes of projection not on the traces is outside of the given plane. Hence follows an easy method to find a line contained in a plane P P′, Fig. 25, when one of its projections ab, is given. The point a where the horizontal projection of the line intersects the horizontal trace α P of the plane, is the trace of the line. Its vertical projection a' is a point of the vertical projection of the line. But the point of intersection, b, of ab with the ground line is the projection of a point of the line AB

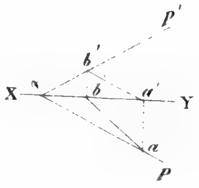

Fig. 25

contained in the vertical plane, that is, the projection of the vertical trace of AB; then if at b a perpendicular bb' be erected to XY, its intersection b' with $\alpha P'$ will be the vertical trace of AB and the vertical projection will be obtained by joining $a'b'$

8. POINT IN A PLANE. When a point M is contained in a plane $P\alpha P'$, Fig.26, one of the projections, m, of the point is sufficient to define it.

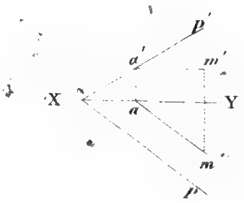

Fig.26

To find the other projection, m', of M, let a horizontal line be drawn through M in the plane $P\alpha P'$, its horizontal projection is a line ma parallel to αP, its vertical trace is at the intersection a' of the vertical trace of the plane $P\alpha P'$ with a perpendicular at a to the ground line and its vertical projection is a line $a'm'$ parallel to XY. The vertical projection of M is then found by drawing through m a perpendicular to XY and producing it to its intersection with $a'm'$. The point mm' being on the line am, $a'm'$, is in the plane $P\alpha P'$.

It is not necessary that the line drawn through M be horizontal; any other line might be employed, but it is more convenient to use a line parallel either to the vertical or to the ground plane.

9. THROUGH A POINT, TO DRAW A PLANE PARALLEL TO ANOTHER PLANE. Let a point mm', Fig. 27, and a plane $P\alpha P'$ be given; through mm', it is required to draw a plane parallel to $P\alpha P'$. Through mm', draw a line parallel to $P\alpha$, its horizontal projection will be a parallel through m to $P\alpha$ and its vertical projection a parallel through m' to the ground line (§ 5). Find the vertical trace a' by erecting at a a perpendicular to the ground line and producing it to its intersection with $m'a'$. Then through a' draw $\beta Q'$ parallel to $\alpha P'$ and through β draw βQ parallel to αP. The plane $Q\beta Q'$ is parallel to the plane $P\alpha P'$ and it contains the point mm', since it contains the line ma, $m'a'$.

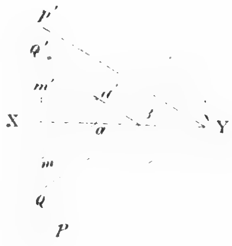

Fig. 27

10. LINE PERPENDICULAR TO A PLANE. Let the line ma, mb', Fig. 28, be perpendicular to the plane $P\alpha P'$: the

projecting plane MA ma is perpendicular to the ground plane: it is also perpendicular to the plane $P\alpha P'$, since it contains a line MA perpendicular to this plane; therefore it is perpendicular to the intersection αP of these two planes and inversely the intersection αP is perpendicular to the projecting plane. But being perpendicular to the plane, it is perpendicular to all lines passing through α and contained in the plane, therefore αP is perpendicular to am.

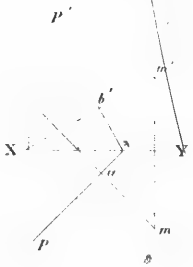

Fig.28

In the same manner it may be shown that $b'm'$ is perpendicular to $\alpha P'$.

So when through a point mm', it is required to draw a line perpendicular to a plane, perpendiculars ma, and $m'b'$ to the traces of the plane are drawn through the projections m and m' of the point.

To draw through a given line a plane perpendicular to a given plane, a line perpendicular to the plane is drawn as explained above from any point of the given line and then a plane is drawn through the two lines by joining the traces of same denomination of the lines.

When it is required to draw through a given point a plane perpendicular to a given line, perpendiculars to the projections of the line are drawn from any point of the ground line: they represent the traces of a plane perpendicular to the given line and there remains only to draw a plane parallel to the first one and passing through the given point as explained in § 9.

11. REVOLVING A PLANE UPON ONE OF THE PLANES OF PROJECTION. For making constructions in a plane other than one of the projection planes, it is often convenient to revolve the plane round one of its traces upon the ground or vertical planes; the construction is then effected and if necessary, the plane is revolved back to its original position.

The problem can always be reduced to finding the position of a point M of a plane $P\alpha P'$, Fig. 29, after this plane has been brought into coincidence with one of the planes of projection, the ground plane for instance, by a revolution round its horizontal trace αP.

From the point m, draw a perpendicular mK to the trace αP of the plane and join MK and Mm the plane MKm is perpendicular to the ground plane as containing Mm; hence it contains the vertical line at K to which αK is perpendicular. But αK is also, by construction, perpendicular [illegible] Km, therefore it is perpendicular to the plane mKM and to KM which is in this plane. Consequently when the plane is revolved round its trace, M will fall on a perpendicular KM_1 to αP.

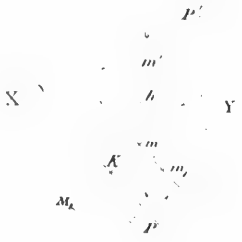

Fig.29

Let us suppose now that the triangle MKm be revolved round Km on to the ground plane; the angle KmM being a right angle, the side mM will fall in mm_1 parallel to αP; mm_1 which is the height of M above the horizontal plane, is equal to hm'. It is therefore easy to construct the triangle and by taking KM_1 equal to Km_1 the position of M_1 is obtained.

The construction lines on the figure have for object merely to show that mm_1 is made equal to km', KM_1 equal to Km_1 and that the point mm' lies in the plane $P\alpha P'$ (§ 8).

A similar construction would be employed to revolve a plane upon the vertical plane.

It may be observed that the angle mKm_1 is the angle of the given plane with the ground plane.

The position of a line revolved upon the horizontal plane is determined by finding the positions of two of its points; its traces for instance. Let $ab, a'b'$, Fig. 30, be the line and $P\alpha P'$ the plane. From b draw a perpendicular bK to αP. The same demonstration as in the case of Fig. 29 will show that αP is perpendicular to the line Kb' in space, therefore b' in its revolution round αP will fall on Kb produced, and KB_1 will be equal to

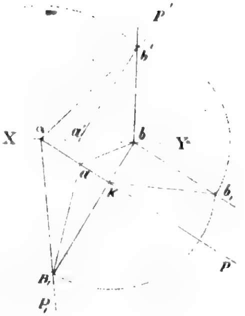

Fig. 30

Kb'. But Kb' is the hypothemuse of the right angle triangle Kbb', which can be constructed at Kbb_1; by making KB_1 equal to Kb_1 the position of B_1 is obtained. Now it must be observed that the position of the horizontal trace a of the given line has not changed; therefore this line after its revolution will fall in αB_1.

Here again the angle bKb_1 is the angle of the given plane with the ground plane and the construction indicated affords a simple method of finding it.

The line αP_1 is the position of the vertical trace after the revolution of the plane; the angle $P\alpha P_1$ is the angle formed in space by the traces of the plane, and αB_1 is equal to $\alpha b'$. Hence the following construction to find the revolved line, when α is within the limits of the drawing.

Draw b K perpendicular to α P and instead of constructing the triangle Kbb_1, describe a circle with α as a center and $\alpha b'$ as radius. Join to a the point of intersection B_1 of the circle with bK produced; $B_1 a$ is the revolved line, and αP_1 the revolved trace of the plane $P\alpha P'$.

To revolve a plane back into its original position, inverse constructions are employed. Let it be required for instance, to find the projections of the point $M_\prime$, Fig.29, when the plane $P\alpha P_\prime$ is revolved back to $P\alpha P'$. The angle of $P\alpha P'$ with the ground plane is first determined by the construction given above: then from $M_\prime$ a perpendicular is drawn to αP and at the point of intersection K, an angle $mKm_\prime$ is constructed equal to the inclination of the given plane on the ground plane. $Km_\prime$ is taken equal to $KM_\prime$ and from $m_\prime$ a perpendicular $m_\prime m$ is drawn to $M_\prime m$; m is the horizontal projection of M, and $mm_\prime$ its height above the ground plane, from which the vertical projection is easily found.

A line is revolved back into its original position by repeating in inverse order the constructions given for revolving it upon the projection plane. Let $aB_\prime$, Fig.30, be the line: from $B_\prime$ and a draw the perpendiculars $B_\prime b$ and aa' to αP and XY respectively. At b erect a perpendicular bb' to XY, produce it to its intersection b' with $\alpha P'$ and join $ab, a'b'$, which are the projections required.

The constructions are simplified when the vertical trace has been revolved on the ground plane. Let it be required to find the position of the point mm', Fig. 31, on the plane $P\alpha P'$ revolved in $P\alpha P_,$ upon the ground plane. From m draw a perpendicular mn to αP: it has been shown that the point M of space in revolving round $\alpha P_,$ will fall upon this line. Through mm' and in the plane $P\alpha P'$ draw a line $ab, a'b'$, cutting the two traces of the plane (§ 7); on αP take $\alpha A_,$ equal to $\alpha a'$ and join $A_,b$. As explained above, $A_,b$ is, in the plane $P\alpha P_,$, the line represented in projection at ab, $a'b'$, and the point required, $M_,$ must be on this line. But it has already been shown that it is on the line mn; therefore it is at the intersection of these two lines, in $M_,$.

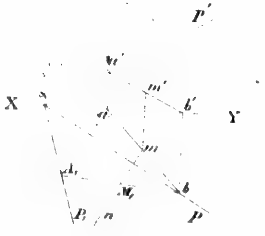

Fig. 31

To revolve back this point into its original position, a line $A_,b$ cutting the traces $\alpha P_,$ and αP is drawn through $M_,$; $\alpha a'$ is taken equal to $\alpha A_,$ and perpendiculars $a'a_{,}$ and bb' are drawn from a' and b to the ground line; ab and $a'b'$ are the projections of the line $bA_{,}$ when revolved back to its original place. A perpendicular to αP is next drawn from $M_{,}$: its intersection with ab gives the horizontal projection m of the point M: the vertical projection is obtained by drawing through m a perpendicular to the ground line and producing to its intersection m' with $a'b'$

Instead of the line ab, $a'b'$, a parallel to the vertical plane may be employed. Let mm', Fig. 32, be the point, $P\alpha P'$ the given plane and $P\alpha P_{,}$ the same plane revolved upon the ground plane. From m draw a perpendicular mn to αP: the point $M_{,}$ will fall on this line. Then through mm' draw ab, $a'b'$ parallel to the vertical trace $\alpha P'$ of the plane (§ 5). When $P\alpha P'$ is revolved, this line will still remain parallel to $\alpha P'$ and as its trace a, does not move, the line will

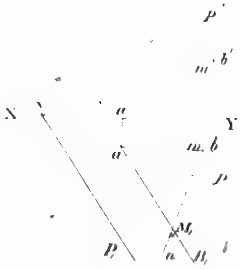

Fig. 32

fall in $aB_,$, parallel to $\alpha P_,$ so the point $M_,$ will be on $aB_,$, but this point is also on mn, therefore it is at the intersection $M_,$ of mn and $aB_,$.

To find the projections of the point M of space when it is given revolved on the ground plane in $M_,$, draw through $M_,$ a parallel $aB_,$ to the trace $\alpha P_,$ and a perpendicular $M_,m$ to αP. Through a draw ab parallel to the ground line; it is the horizontal projection of a line parallel to the vertical plane and passing through the point M. But the horizontal projection m of M is also on the line $M_,m$, therefore it is at the intersection of $M_,m$ and ab.

The vertical projection m' of M is on the perpendicular mm' drawn through m to the ground line; it is also on the vertical projection $a'b'$ of the parallel to the vertical plane, which is obtained by drawing from a a perpendicular aa' to XY and through a' a parallel to the trace $\alpha P'$. The intersection of ab' and mm' gives the vertical projection m' of M.

The constructions are still further simplified when the given plane is perpendicular to one of the planes of projection.

Let $P\alpha P'$, Fig. 33, be a plane perpendicular to the ground plane and mm' a point of the plane. The point M in space is on the vertical line passing through m, which line is in the plane $P\alpha P'$ and is perpendicular to the horizontal line αP. Therefore, when $P\alpha P'$ is revolved round αP, the line mM will still remain perpendicular to αP and the point M will fall in M_1 at a distance mM_1 from αP equal to the height of M above the ground plane. But this height is hm'; therefore to determine the point M_1, draw at m a perpendicular to αP and take mM_1 equal to hm'.

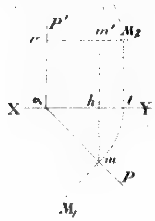

Fig. 33

Instead of revolving the plane round αP, it may be revolved round $\alpha P'$ on the vertical plane. The point M will then describe in space an arc of circle of which the vertical projection is the line $m'M_2$ parallel to XY and the horizontal projection an arc of circle mt, with α as a center and αm as radius. When the plane $P\alpha P'$ coincides with the vertical plane, the point M of the plane must be somewhere on the line $m'M_2$ and its horizontal projection is t. Then if a perpendicular to XY be erected at t and produced to its intersection M_2 with $m'M_2$, M_2 will be the required point.

To find the projections of the point M whose position M_1 revolved on the ground plane is given, draw from M_1 a perpendicular M_1m to αP and from m a perpendicular mm' to XY; take hm' equal to mM_1, height of the point M above the ground plane; m, m' are the projections of the point.

The projections of M_2 are found by drawing through M_2 a parallel M_2v to XY, which is the vertical projection of the arc of circle described by M_2 when revolved back to its original position; take $\alpha m'$ equal to the distance M_2v of M_2 from the trace $\alpha P'$ and through m draw the perpendicular mm' to XY m, m' are the projections of the point.

12. INTERSECTION OF TWO PLANES. Let $P\alpha P'$ and $Q\beta Q'$, Fig.34, be two planes: the points M and N where the traces of the planes meet, are the traces of the line of intersection of the planes. The projections

Mn, m'N of the intersection are found as explained in § 4 by letting fall the perpendiculars Mm' and Nn to the ground line and joining Mn, m'N.

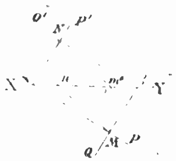

Fig.34

13. THE INTERSECTING PLANES ARE BOTH PARALLEL TO THE GROUND LINE. Let PQ, P'Q', RS, R'S', Fig.35, be the traces of two intersecting planes parallel to the ground line: the construction given in § 12 does not apply and recourse must be had to an auxiliary plane. Draw a plane TOT' perpendicular to the ground line. The line of intersection of the two given planes is parallel to the ground line and so are its projections. If the projections m and m' of the point M where this line intersects the plane TOT' were known, the projections of the line itself would be obtained at once by drawing through m and m' parallels to the

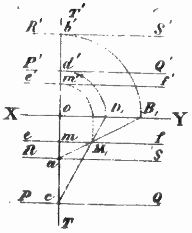

Fig.35

ground line.

To obtain M, let us revolve TOT$'$ around OT upon the ground plane: the intersection of TOT$'$ and PQ P$'$Q$'$; of which the traces are c and d', will fall in cD_1; OD$_1$ being equal to Od'. Similarly the intersection of TOT$'$ and RS R$'$S$'$ will fall in aB_1, OB$_1$ being equal to Ob' and the point M will come in M$_1$, at the intersection of cD_1 and aB_1. From M$_1$ draw a perpendicular to OT; the point of intersection m is the horizontal projection of M. The vertical projection is obtained by making Om' equal to mM$_1$, this being the height of M above the ground plane. Then through m and m' draw the parallels ef, $e'f'$, to the ground line; they are the projections of the line of intersection.

14. THE INTERSECTING PLANES CUT THE GROUND LINE AT THE SAME POINT. Let PαP$'$, QαQ$'$, Fig. 36, be two intersecting planes cutting the ground line at α. Draw a plane TOT$'$ perpendicular to this line; α is a point of the line of intersection of the planes, and if the projections m and m' of the point M where this line cuts the plane TOT$'$ were known, the projections of the intersection would be obtained by joining αm and $\alpha m'$.

Let us revolve the plane TOT' around OT: the intersection of TOT' and $P\alpha P'$, of which a and b' are the traces, will fall in aB_1, OB_1 being equal to Ob'. Similarly the intersection of TOT' and $Q\alpha Q'$ will fall in cD_1, OD_1 being equal to Od', and M will come in M_1, at the intersection of aB_1 and cD_1. From M_1, draw a perpendicular M_1m to OT, the point m is the horizontal projection of M. The vertical projection m' is obtained by making Om' equal to mM_1. Then draw αm and $\alpha m'$ which are the projections of the intersection.

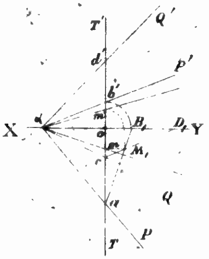

Fig.36

15. INTERSECTION OF TWO PLANES, ONE OF WHICH IS HORIZONTAL OR PARALLEL TO THE VERTICAL PLANE. When one the planes is horizontal, the intersection is parallel to the horizontal trace of the other plane: its vertical projection is the trace $Q'R'$, of the horizontal plane (Fig.37) and the horizontal

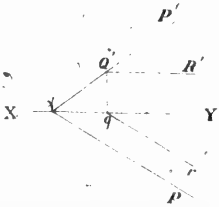

Fig.37

projection a parallel qr to αP.

In the case of a plane parallel to the vertical plane (Fig.38), the horizontal projection of the intersection is the trace QR of the vertical plane. The vertical projection is a parallel $q's'$ to the vertical trace $\alpha P'$ of the other plane.

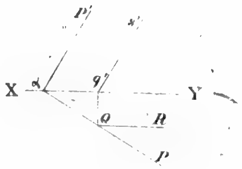

Fig.38

16. PLANES PERPENDICULAR TO ONE OF THE PLANES OF PROJECTION. When the two planes are both perpendicular to one of the planes of projection, their intersection is also perpendicular to this plane and its projection on it is the point where the traces of the planes meet. The projection on the other plane is a perpendicular to the ground line passing through the above point.

17. INTERSECTION OF A LINE AND A PLANE. To find the intersection of a line and a plane, another line intersecting the first one is drawn in the plane; the point required is the intersection of the two lines.

Let *ab, a'b',* Fig.39, be the line and $P\alpha P'$ the plane. For auxiliary line, the intersection of $P\alpha P'$ by one of the projecting planes of the given line, *abb',* for instance, may be employed.

Fig.39

To obtain the projections of this intersection, draw the perpendicular *bb''* and *cc'* to the ground line and join *c'b''*; *cb, c'b''* is the intersection. It meets the line *ab, a'b'* at *mm'* which is the point where the line cuts the plane $P\alpha P'$.

18. INTERSECTION OF THREE PLANES. The intersection of three planes may be found either by constructing the line of intersection of two of the planes and then determining the point where this line cuts the third plane or by constructing the lines of intersection of one of the planes with each of the others: the point where the two lines meet is the point of intersection of the three planes.

19. THROUGH A POINT, TO DRAW A STRAIGHT LINE WHICH WILL MEET TWO GIVEN LINES. To draw through a point

a straight line which will meet two given lines not in the same plane, a plane is passed through the point and one of the lines. The point where the second line pierces the plane is ascertained (§ 17) and by joining this point of intersection to the given point, the line required is obtained.

20. **DISTANCE OF TWO POINTS.** Let aa', bb', Fig. 40, be two points; to obtain their distance, one of the projecting planes of the line AB may be revolved about its trace upon the corresponding projection plane.

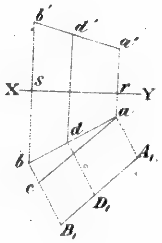

Fig. 40

Let us revolve, for instance AB ab around ab. The point A will fall in $A_,$ on a perpendicular $aA_,$ to ab, the line $aA_,$ being the height of A above the ground plane, that is the distance ra'. Similarly B will fall in $B_,$, on a perpendicular $bB_,$ to ab, and at a distance from b equal to sb'. The required distance of the points is $A_,$ $B_,$.

The construction may be somewhat simplified by observing that if a line be drawn through a parallel to $A_{,}B_{,}$, its length ac is equal to $A_{,}B_{,}$; therefore instead of constructing the trapezoid $aA_{,}B_{,}b$, it will be sufficient to erect a perpendicular to ab at b and to lay off on it a distance bc equal to the difference between sb' and ra'.

21. TO LAY OFF A GIVEN LENGTH ON A LINE. The construction given in § 20 may be employed for laying off a given length on a line AB (Fig. 40). Turn the projecting plane on the line ab as an axis and lay off the required length $A_{,}D_{,}$ on $A_{,}B_{,}$. Then revolve the projecting plane back to its natural position: the horizontal projection of D will be at d, foot of the perpendicular drawn from $D_{,}$ to ab, and its vertical projection will be at d', intersection of $a'b'$ by a perpendicular through d to the ground line.

22. DISTANCE FROM A POINT TO A LINE. The distance from a point to a straight line is obtained by passing a plane through the line and the point, and revolving it upon one of the planes of projection. Let $ab, a'b'$, be the line and mm' the point (Fig. 41). Through mm' draw a parallel $cd, c'd'$ to $ab, a'b'$: the line ac is the horizontal trace of the plane

containing the two parallel lines. Revolve this plane around its trace ac, until it coincides with the ground plane (§ 11).

Let $aB_{,}$ and $M_{,}$ be the revolved positions of ab' and M. From $M_{,}$ let fall a perpendicular $M_{,}K$ to $aB_{,}$; it is the distance required.

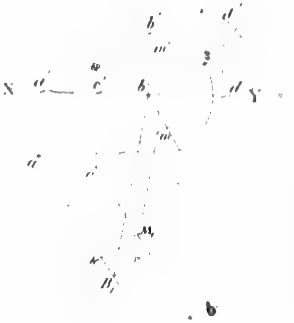

Fig.41

23. DISTANCE FROM A POINT TO A PLANE. The distance from a point to a plane may be obtained by dropping a perpendicular from the point to the plane (§ 10), finding the point where it pierces the plane (§ 17) and determining the distance of the two points.

It is more convenient to pass through the point a plane perpendicular to one of the traces of the given plane. This auxiliary plane, being perpendicular to the other one, contains the perpendicular from the point to the given plane: by revolving it around its trace upon one of the planes of projection, the following simple construction gives at once the solution of the problem.

Let $P\alpha P'$, Fig.42, be the plane and mm' the point.

Fig.42

Through mm' pass the plane $Q\beta Q'$ perpendicular to $\alpha P'$ and revolve it around βQ upon the ground plane. The point A describes the arc of circle AA_1, and BA_1 is the intersection of the two planes revolved upon the ground plane.

The point M is on a parallel to βQ passing through m'. In revolving the auxiliary plane, m' describes the arc of circle $m'c$ and the line $m'M$ falls in cM_1, still parallel to βQ. The point M remaining during the revolution of the plane at a constant distance from the vertical plane, will fall on a parallel to the ground line passing through m; therefore M_1 will come at the intersection of cM_1 and mM_1. There remains only to let fall a perpendicular from M_1 to BA_1: it is the distance required.

24. **DISTANCE OF TWO PARALLEL PLANES.** The distance of two parallel planes may be obtained by intersecting

them by a third plane perpendicular to both and revolving it upon one of the planes of projection.

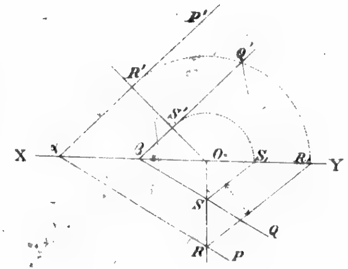

Fig.43

Let $P\alpha P'$, $Q\beta Q'$ Fig.43 be the parallel planes. Draw a plane ROR' perpendicular to the vertical traces and revolve it upon the ground plane around OR as an axis. The points R' and S' describe the arcs of circle $R'R_1$, $S'S_1$:the lines RR_1 and SS_1 being the intersections of the given planes by the auxiliary one. These lines are parallel and their distance is the distance of the planes.

25. DISTANCE OF TWO STRAIGHT LINES. Let AB and CD Fig.44 be two straight lines not contained in one plane;it is required to find their shortest distance. This distance is the perpendicular to both lines. Through any point of AB, A for instance,draw a parallel AF to CD and from a point G of CD,let fall a perpendicular GH on the plane BAF. Through the foot of GH in the plane BAF,draw a parallel HK to AF and

through K another parallel KM to HG. the line KM is perpendicular to both lines.

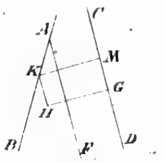

Fig.44

Although presenting no difficulty the construction requires many lines and is omitted here.

26. **ANGLE OF A LINE WITH THE PLANES OF PROJECTION.** Let it be required to find the angles formed by the line $ab, a'b'$, Fig.45, with the planes of projection.

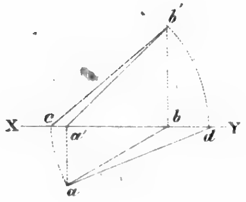

Fig.45

The angle of the line with the ground plane is the same as with the line ab, since the plane $b'ab$ is perpendicular to the ground plane. This angle can be obtained by revolving the triangle $b'ba$ around $b'b$ as an axis upon the vertical plane. The vertex a describes the arc of circle ac and the triangle comes in $b'bc$ the angle at c being the angle of the line with the ground plane.

Similarly the angle with the vertical plane is obtained by revolving the triangle aab upon the ground plane around aa' as an axis. The vertex b' comes in d, the angle ada' being the angle of the line with the vertical plane.

When the line is contained in a plane perpendicular to the ground line, such as ab', Fig.46, the angles are found by revolving the plane upon one of the planes of projection, the ground plane for instance: the vertical trace, b' describes the arc of circle $b'B_1$ and the revolved position of the line is aB_1; a and B_1 are the angles with the ground and vertical planes respectively.

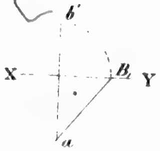

Fig.46

In the case of a line parallel to one of the planes of projection, the angle of the line with the other plane is the angle of its projection with the ground line.

27. ANGLE OF TWO LINES. To find the angle formed by two intersecting lines, their plane is revolved

about its trace upon one of the planes of projection. Let *ab, a'b'; cd, c'd'*, Fig.47, be the lines. The horizontal trace of their plane is the line *ac* passing through the traces; it forms with the two lines a triangle *a*M*c*, in which M is the angle to be found.

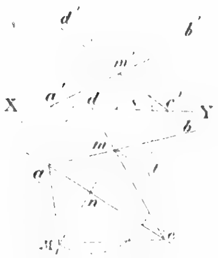

Fig.47

Revolve this triangle around *ac* upon the ground plane; the point M will move in the plane perpendicular to *ac* whose horizontal trace is the perpendicular *mn* to *ac*; it will therefore fall in $M_{,}$, somewhere on *mn* produced. The distance *n*$M_{,}$ is the same as ~~the~~ distance from *n* to M and the latter is the hypothenuse of the right angle triangle M*mn*. But the side M*m* of this triangle is the height of M above the ground line and the triangle can be constructed by erecting at *m* a perpendicular to *mn* and laying off *mt* equal to *m's*; $M_{,}$ is then determined by making *n*$M_{,}$ equal to *nt*. Joining $M_{,}$*a* and $M_{,}$*c*, the angle required is *a*$M_{,}$*c*.

It may happen that the traces of the lines are outside of the drawing, and that the trace of their plane can not be obtained as explained above. In that case, the lines are cut by an auxiliary horizontal plane on which the construction of Fig. 47 is effected.

When the lines are parallel to one of the planes of projection their angle is the angle of their projections on that plane.

28. ANGLES OF A PLANE WITH THE PLANES OF PROJECTION.

The angles of a plane with the planes of projection are obtained by cutting it by auxiliary planes perpendicular to the traces. Let $P\alpha P'$, Fig. 48, be the plane. Draw a plane TOT' perpendicular to αP: its intersections with the planes of projection and the given plane form a right angled triangle TOT' in which the angle at T is the angle of $P\alpha P'$ with the ground plane. Revolve this triangle around

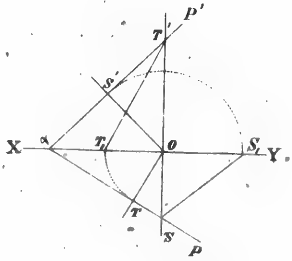

Fig. 48

OT' as an axis upon the vertical plane: T describes an arc of circle TT_1, of which O is the center and the triangle comes in $T'OT_1$, the angle T_1 being the angle of $P\alpha P'$ with the ground plane.

Similarly, the angle with the vertical plane is obtained by drawing the plane SOS' perpendicular to $\alpha P'$ and revolving the triangle SOS' upon the ground plane in SOS_1. The angle at S_1 is the angle of $P\alpha P'$ with the vertical plane.

The line $T'T$ is the line of greatest declivity of the plane $P\alpha P'$: any other line contained in the plane $P\alpha P'$ and not parallel to $T'T$ forms with the ground plane an angle smaller than $T'TO$.

29. ANGLE OF TWO PLANES. Let $P\alpha P'$, $Q\beta Q'$, Fig. 49, be two planes, of which it is required to find the angle. Their intersection is projected horizontally in *ab*. Cut the planes by another one perpendicular to both; it is perpendicular to their intersection and consequently the horizontal trace *cd* is perpendicular to *ab*. The intersections of this plane with the two given planes form with the trace *cd* a triangle in which the angle opposite *cd* is the angle of the two planes.

The intersection of the auxiliary plane with

Fig.49

the projecting plane *abb′* is the perpendicular let fall from the vertex of the triangle on *cd* because *cd* being perpendicular to the projecting plane is perpendicular to all lines contained in it passing through its foot K. The same intersection is also perpendicular to the intersection *ab′* of the two given planes, because *ab′* being perpendicular to the auxiliary plane, is perpendicular to all lines contained in that plane by which it is intersected.

Now revolve the triangle *abb′* about its side *ab* upon the ground plane. The angle at *b* being a right angle, the point *b* will fall in B, on a perpendicular to *ab* at *b*, *b*B, being equal to *bb′*. Join B,*a* and let fall on it from K a perpendicular KH, : this is the height of the triangle formed by *cd* and the intersections of the two given planes by the auxiliary plane. Then revolve this triangle around *cd* upon the ground plane;its vertex will fall on the line *ab* at a distance K*h* equal to KH ;join *hc*, *hd* and *chd* is the angle required.

When the planes are in such a position as to make the above construction inconvenient,they may be replaced by parallel planes,whose positions are selected at pleasure. This may be done,for instance, when the planes cut the ground line at the same point or when their traces do not meet within the limits of the drawing.

When the planes are both parallel to the ground line,the construction is the same as in Fig.35; *aMc* is the angle of the planes.

30. THROUGH A GIVEN LINE IN A PLANE TO DRAW ANOTHER PLANE MAKING A CERTAIN ANGLE WITH THE GIVEN PLANE. The converse problem consists in drawing through a given line of a plane,another plane making with the first one a given angle. The construction is the same as in Fig.49,but is inverted. The given line is the intersection of the two planes:the triangle *chd*

is constructed by means of the line KH, and the angle *h*, it gives a point *d* of the horizontal trace of the plane required. Another point of the trace is found at *a*, then join *ad*, produce to *β* and join *βb′*: the required plane is *aβb′*.

31. ANGLE OF A LINE WITH A PLANE. The angle of a line with a plane is the complement of the angle of the line with a perpendicular to the plane. So in order to find the first angle, a perpendicular may be erected to the plane through a point of the given line (§ 10); the angle of the two lines is then determined (§ 27).

32. METHOD OF ROTATIONS. The method of rotations is a process employed in Descriptive Geometry for facilitating the solution of problems. It consists in rotating the whole system of the projections or only part of it, around an axis perpendicular to one of the planes of projection, until the system assumes a position favourable to the solution of the problem.

33. ROTATION OF A POINT. Let it be required to rotate a point *mm′*, Fig. 50, through an angle *a*, around a vertical axis *a, a′b′*. The projection *m* will describe

an arc of circle mm_1 with center at a and subtending an angle equal to ω. But the point M, during its motion remains at the same distance from the ground plane; therefore its vertical projection, m', travels on a parallel $m'm'_1$ to the ground line. So when the point m has described the arc ω, the point m is in m'_1, at the intersection of the perpendicular to the ground line through m_1 with the parallel to the same line through m'.

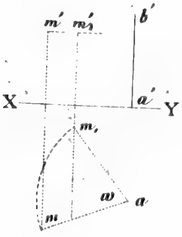

Fig. 50

34. ROTATION OF A LINE. Let $ab, a'b'$, Fig. 51, be a straight line to be rotated around a vertical axis $c, c'd'$ until parallel to the vertical plane. From c let fall the perpendicular cm on ab, and rotate the projecting plane containing ab around the axis. The point m will describe an arc of circle and stop at m_1 on the perpendicular to

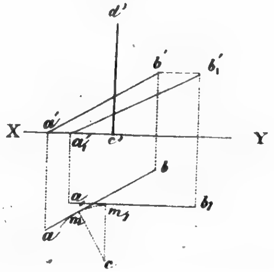

Fig. 51

the ground line drawn through c. The projecting plane will then be parallel to the vertical plane and so will the lines ab and AB. The new position of ab is obtained by drawing through m_1 a parallel a_1b_1 to the ground line and making $a_1m_1 = am$; $b_1m_1 = bm$. In their motion around the axis, the height above the ground plane of the points A and B of the given line does not change; the vertical projection a' of A, will therefore move on the ground line and the projection b' on a parallel $b'b'_1$ to the ground line. But a'_1, the new vertical projection of A, must be on the perpendicular through a_1 to the ground line and since it is also on the ground line, it must be at their intersection in a'_1. Similarly, b'_1 must be on the perpendicular $b'b'_1$ to the ground line and also on the parallel $b'b'_1$, therefore it must be at their intersection b'_1. The rotated vertical projection is then $a'_1b'_1$

35. ROTATION OF A PLANE. A plane may be rotated by turning three of its points, not on a straight line (§ 33), or a point and straight line, both in the plane, or two of its lines (§ 34). The following method is a simple one.

$P\alpha P'$, Fig. 52, is a plane to be rotated until perpendicular to the vertical plane, about a vertical axis of which the horizontal trace is at c. From c let fall a perpendicular cd on αP and rotate αP until cd is parallel to the ground line: αP_1 will then be perpendicular to XY. It is the rotated horizontal trace of the plane.

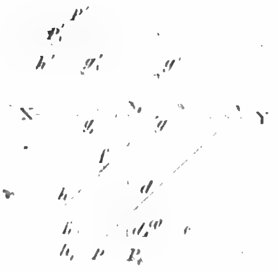

Fig. 52

Now draw any horizontal line $gh, g'h'$, in the plane $P\alpha P'$; produce cd to its intersection f with gh and rotate the line gh, $g'h'$, through the same angle, ω, as the trace αP of the plane. The point f of ab will describe the arc of circle ff_1 and stop on cd_1 produced. The rotated horizontal projection will then be a line g_1h_1 perpendicular to XY.

To obtain the vertical projection, it must be observed that the height of $gh, g'h'$, above the ground plane is gg' and that it does not change during the rotation. The vertical trace g' will then move on the parallel $g'h'$ to XY, and will stop at the intersection of $g'h'$ and g_1h_1 produced, since g_1h_1 is perpendicular to XY.

The rotated line $g_1h_1g_1'$, is still parallel to the ground plane and is now contained in a plane perpendicular to the vertical plane; therefore it is itself perpendicular to the vertical plane. Its vertical projection is the point g_1', which is also its trace and consequently a point of the vertical trace of the rotated plane. But α_1 is another point of the new vertical trace, therefore the rotated plane is $P_1\alpha_1P_1'$.

The angle $g_1\alpha_1g_1'$ is the inclination of the rotated plane on the ground plane: this inclination is the same before and after rotation.

The plane might now be brought parallel to the ground plane by a second rotation about an axis perpendicular to the vertical plane.

36. DISTANCE OF TWO POINTS. As an application of this method, the determination of the distance of two points may be given.

Let aa', bb, Fig. 53, be the points. Rotate the vertical projecting plane containing a and b around the vertical line through a until it is parallel to the vertical plane. The point b' describes an arc of circle bb_1 and stops at b_1 on the parallel ab_1 to XY; b' moves on a parallel to XY and stops at b_1' at the intersection of the parallel $b'b_1'$ and the perpendicular b_1b_1', to the ground line. The rotated line is now parallel to the vertical plane; it is therefore equal to its vertical projection $a'b_1'$. The inclination of the line on the ground plane is that of the vertical projection on the ground line.

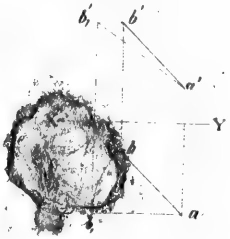

Fig. 53

Another solution of this problem is given in § 20.

37. SOLUTION OF SPHERICAL TRIANGLES. A spherical triangle may be assimilated to a trihedral angle by supposing the vertex of the angle to be at the center of the sphere. The sides of the spherical triangle are then subtended by the plane angles of the faces of the trihedral angle and the angles of the triangle are the same as the dihedral angles of the trihedral angle.

As usual, the sides of the spherical triangle are designated by *a*, *b* and *c*, the opposite angles being A, B, and C.

38. GIVEN THREE SIDES TO FIND THE ANGLES. The three sides of the triangle correspond to the three faces of the trihedral angle. Develop them on the ground plane, placing one of the edges, OQ, Fig. 54, perpendicular to the ground line and revolving the faces *a* and *c* about the edges OQ and OR, upon the ground plane. The intersection of the trihedral angle by the vertical plane forms a pyramid of which O is the vertex and OQR one of the faces in its natural position. Since OQ is perpendicular to the vertical plane, the planes of the two faces intersecting along OQ are also perpendicular to the vertical plane, therefore OQP, is one of the faces of the pyramid, revolved upon the ground plane about OQ, and OP is the third edge of the

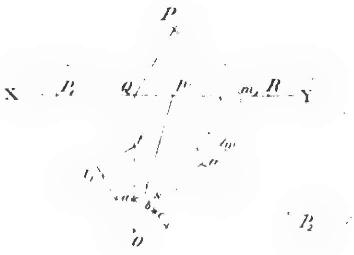

Fig. 54

pyramid, the vertical trace of which is on the arc of circle described from Q as a center with QP_1 as radius.

The third edge of the pyramid is also shown in OP_2 which must be taken equal to OP_1 ; P_2 , like P_1 , is the vertical trace of the third edge OP revolved upon the ground plane. Let now the face *c* be revolved back to its natural position, by turning it about OR: the horizontal projection of P_2 will move on the perperpendicular P_2m, let fall from P_2 on OR, and when P_2 comes to its original place in the vertical plane, its horizontal projection will have moved along P_2m up to its intersection *p* with the ground line. The vertical trace P will therefore be on the perpendicular *p*P to the ground line, but being also on the arc of circle P_1 P, it is at their intersection.

Having now obtained the trace P of the edge OP on the vertical plane, the dihedral angle C is found at once in PQR, since both faces are pendendicular to the vertical plane.

Generally, only one angle is required: in making the construction, the edge corresponding to this angle is placed perpendicular to the ground line.

Should the other angles be wanted, A could be obtained from the triangle pmP revolved around Pp on the vertical plane; Pm_1p is the angle A of the spherical triangle. B is constructed as explained in § 29 or by any other method.

39. GIVEN TWO SIDES AND THE INCLUDED ANGLE, TO FIND THE REMAINING SIDE AND ANGLES. Let a, b and C, be given; required c, A and B.

Place the intersection of a and b in OQ, Fig. 54, perpendicular to XY, and the face b on the ground plane; draw QP making the angle PQY equal to C: QP is the vertical trace of the face a. Make the angle QOP_1 equal to a : QOP_1 is the face a of the trihedral angle revolved about OQ on the ground plane. Taking QP equal to QP_1, the point P is the vertical trace of the third edge of the trihedral angle.

To obtain c, let fall from P and p the perpendiculars Pp and pm to XY and OR respectively. Revolve about Pp on the vertical plane the triangle formed in space by Pp and pm : Pm_1p is the angle A.

Then produce pm and take mP_2 equal to m_1P: join

OP_2 : ROP_2 is *c*. B is obtained as explained in § 38.

40. GIVEN TWO ANGLES AND THE SIDE OPPOSITE ONE OF THEM, TO FIND THE REMAINING SIDES AND ANGLE. Let *a*, A, and B be given: required C, *b* and *c*.

Place the face *c* on the ground plane and the intersection of *a* and *c* in OP, Fig. 55, perpendicular to the ground line. Through P draw PQ making with XY the angle B; PQ is the vertical trace of the face *a*, since *a* and *c* are both perpendicular to the vertical plane. Draw OQ_1 making the angle *a* with OP; POQ_1 is the face *a* of the trihedral angle revolved upon the ground plane about OP as an axis, Q_1 is the revolved vertical trace of the edge OQ. Making then PQ equal to PQ_1 gives the trace Q. Through Q draw Qm_1 making the angle A with XY, and let fall the perpendicular Qq to XY. From q as a center and qm_1 as

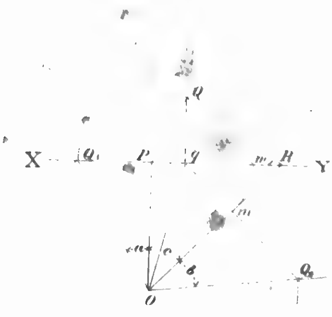

Fig. 55

radius, describe the arc of circle mm_1 and through O draw the tangent OR to the circle. The angle of the plane ORQ with the ground plane is equal to A, therefore ROQ is the face *b* of the trihedral angle and POR is the face *c*.

b and the angle C are obtained as in former cases.

41. GIVEN TWO SIDES AND THE ANGLE OPPOSITE ONE OF THEM, TO FIND THE REMAINING SIDE AND ANGLES. Let *a*, *b* and B be given: required A, C and *c*.

Place the face *a* on the ground plane with the intersection of *a* and *b* in OP, Fig. 56, perpendicular to XY. Make POR and POQ_1 equal to *a* and *b* respectively: POQ_1 is the face *b* of the trihedral angle revolved about OP on the ground plane, therefore the vertical trace Q of the edge opposite to *a* is on a circle described from P as a center with PQ_1 as radius. Through P pass a plane perpendicular to OR: its

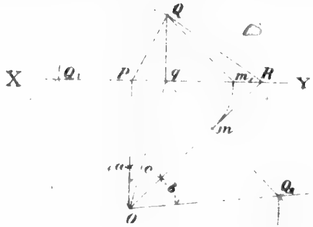

Fig. 56

horizontal trace is a line Pm perpendicular to OR and its vertical trace the perpendicular PS to XY. The intersections of this plane with the two planes of projection and the plane of the face c, form in space a triangle SPm in which P is a right angle and m is the angle B. Revolving this triangle about SP upon the vertical plane, in SPm_1, the point S is obtained. But S is a point of the vertical plane of projection and is also a point of the plane of the face a, therefore it is a point of the trace of the last plane. Joining then RS, the intersection of this line with the circle Q_1Q is the vertical trace of the edge of the trihedral angle opposite to a.

A, C and c are now constructed as in former cases.

42. OTHER CASES--SUPPLEMENTARY TRIANGLES. The other cases of spherical triangles are generally solved by the use of the supplementary triangles or trihedral angles. The direct solution, although possible, is not so convenient. The angles A_1, B_1, C_1, of the supplementary triangle are the supplements of the sides a, b, c, of the other triangle and the sides a_1, b_1, c_1,

of the supplementary triangle are the supplements of the angles A, B, C, of the other one.

From any point O, Fig. 57, in the interior of the trihedral angle, let fall perpendiculars OS, OT, OV, on the faces. The angle of OT and OV is the supplement of the angle of the planes to which they are perpendicular. But the angle of the planes is the angle B of the trihedral angle; therefore TOV or b_1 is equal to 180° – B.

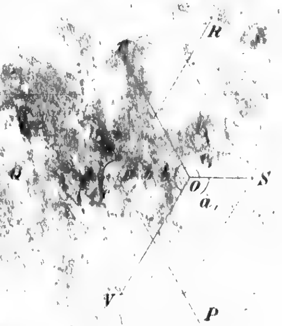

Fig. 57

Similarly: $TOS = c_1 = 180° - C$

$SOV = a_1 = 180° - A.$

The plane TOV containing perpendiculars to a and c, is perpendicular to both; therefore it is perpendicular to their intersection DQ [illegible] conversely DQ is perpendicular to TOV. For the same reasons DR is perpendicular to TOS and DP to VOS. Therefore the angle of DQ and DR or a, is the supplement of the angle formed by VOT and TOS or $A_{,}$;

$$A = 180° - a.$$

In the same manner, it may be shown that:

$$B_{,} = 180° - b,$$

and $$C_{,} = 180° - c,$$

Hence the trihedral angle QTSV is the supplementary angle of DPQR.

43. REDUCTION OF AN ANGLE TO THE HORIZON. The reduction of an angle to the horizon is an application of the solution of spherical triangles. When an angle is observed between two points which are not in the horizontal plane of the observer, the observed angle requires a correction to reduce it to the angle formed by the projections of the points on the ground plane. For that purpose the observer measures the angular elevations or depressions of the points.

Take as vertical plane of projection the plane passing through the observer and one of the points. Assume any point P, Fig. 58, as the place of observation and draw through it the lines PA and PB, making with the ground line angles equal to the elevations or depressions α and β of A and B.

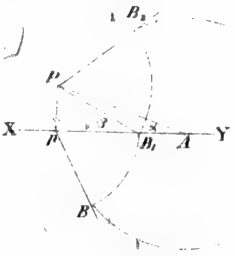

Fig. 58

The lines PA, PB and the vertical Pp form a trihedral angle in which the faces are $90° - \alpha$, $90° - \beta$, and the observed angle. A pyramid is cut off this trihedral angle by the ground plane, the base of the pyramid being the triangle pAB, in which pA and pB are two sides and p the observed angle reduced to the horizon. The third side may be found by revolving the face APB of the pyramid around AP, upon the vertical plane. This face will come in APB_2, the angle at P being the observed angle and

$$PB_2 = PB = PB_1$$

We have now the third side of the triangle A B :hence describing arcs of circles from p and A as centers with AB_2 and pB_1 as radius respectively, their intersection is the point B and ApB is the required angle.

CHAPTER II

PERSPECTIVE

44. Perspective is that part of Geometry which treats of the representation by figures drawn on a surface of objects placed beyond it. Generally this surface is a vertical plane;it is called "picture plane" The figures drawn on it,according to the rules of perspective,produce on the eye as far as form is concerned, the same impression as the objects themselves seen in their actual places.

Suppose a transparent plane surface,such as glass,placed between the eye and the objects to be represented. If the outlines of the objects seen through the glass could be traced on it,the image thus formed would be an exact perspective.

Considering a visual ray from the eye to a point of space:this ray pierces the picture plane in a second point,which is called the "perspective" of the first one.

The visual rays from the eye to all the points of a straight line form a plane whose intersection with the picture plane is the perspective of the line. Consequently,the perspective of a straight line is another straight line.

When the line is a curve,the visual rays to its various points form a conic surface whose vertex is at the eye and whose intersection with the picture plane is the perspective of the curve. A surface of the same nature is formed by the visual rays tangent to the visible outline of an object:the perspective of the object is the intersection of this surface by the picture plane.

45. DEFINITIONS. The "ground plan" is the horizontal projection of the objects to be represented; thus for the perspective of a landscape,the ground plan is the topographical plan of the ground;for a building,it is the horizontal or ground plan of the

building (ABCD,Fig.59).

The "ground plane" is the plane on which the ground plan is placed (KX s Y,Fig.59). For a landscape,it may be,for instance,the horizontal plane passing through the datum point of the topographical plan and for a building,the basement or first floor plane. Any horizontal plane may,however,be used as ground plane,provided its altitude be taken into account:the ground plan does not change,whatever the altitude may be.

The "elevation" is the vertical projection of an object:the elevations of a building are those plans of the building which show the front,rear,or sides.

The "picture plane",as already explained,is the plane on which the perspective is drawn (FFXY,Fig.59). Generally,it is vertical and placed between the eye and the object to be represented,but none of these rules is absolute. Perspectives are sometimes drawn on planes which are not vertical and objects are represented which are between the picture plane and the eye. Such a position of objects is the rule and not the exception,in perspectives used for surveying, when they are taken as representations not of the ground itself, but of a model of it reduced to the scale of the map. This convention will be found further on. Objects are even represented which are behind the observer, the origin of light, for instance in the construction of shadows, but this is merely a geometrical conception to which the usual definition of a perspective does not apply.

The "ground line" is the intersection of the ground and picture planes (XY, Fig. 59).

The "station" is the point supposed to be occupied by the eye of the observer. (S, Fig. 59)

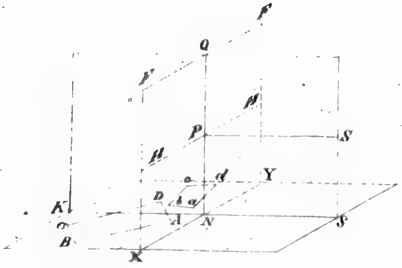

Fig. 59

The "foot of the station" is the point where the vertical of the station pierces the ground plane (*S* , Fig. 59).

The "principal point" is the foot of the perpendicular drawn from the station to the picture plane;

it is shown in P, Fig. 59.

The "distance line" is the line between the station and the principal point (SP, Fig. 59). Its length is the distance from the station or from the foot of the station to the picture plane.

The "horizon plane" is the horizontal plane passing through the station. It contains the distance line and cuts the picture plane on a horizontal line passing through the principal point and called "Horizon line" (HH, Fig. 59). The distance between the horizon line or the principal point and the ground line is equal to the altitude of the station.

The "principal plane" is the vertical plane perpendicular to the picture plane and passing through the station (SNQ, Fig. 59). It contains the foot of the station, the principal point and the distance line.

The "principal line" is the intersection of the principal and picture planes (QN, Fig. 59). It is perpendicular to the ground and horizon lines and intersects the latter at the principal point.

A "front plane" is a plane parallel to the picture plane.

A "front line" is any line contained in a front plane, therefore any line parallel to the picture plane.

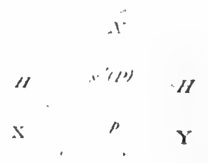

Fig.60

In Fig.60, these points, lines and planes are represented by their orthogonal projections: the ground plane is taken for horizontal plane and the picture plane and for vertical plane. ss' is the station, s the foot of the station, s' or P the principal point HH' the horizon line, sp, s' the distance line and sp N the principal plane.

46 PERSPECTIVE OF A POINT IN THE GROUND PLANE. Let XYs, Fig.61, be the ground plane, XYN the picture plane, S the station and M a point in the ground plane. The perspective of M on the picture plane is the point where the straight line SM pierces the picture plane, that is the vertical trace of SM, M being the

Fig.61

horizontal trace. Thus we have the first relation between a point of the ground plane and its perspective, they are the traces of the visual ray on the ground and picture planes respectively.

6

Fig. 62

Fig. 62 represents in orthogonal projection the construction of Fig. 61; *ss'* is the station, M the point of the ground plane *sa, m's'*, the visual ray and *μ* the perspective of M. The points M and *μ* are the traces of *sa, m's'*.

47. PERSPECTIVE OF A LINE IN THE GROUND PLANE. It has been shown in § 43 that the perspective of a straight line is the intersection with the picture plane of the plane containing the station and the given line.

Fig. 63

Draw a plane through the straight line AB and the station S, Fig. 63. The intersection *αβ* of this plane with the picture plane is the perspective of AB. Thus we have this relation between a straight line in the ground plane and its perspective: they are the traces on the ground and picture planes, of the plane containing the station and the line itself.

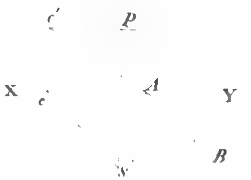

Fig. 64

In orthogonal projection, the line being in the ground plane, the horizontal projection is the line itself, AB, Fig. 64, the vertical projection is the ground line. To draw a plane through the station sP, and the line AB, draw through sP a parallel to AB; the horizontal projection is a parallel to AB through s, and the vertical projection a parallel through P to the ground line. The vertical trace is at c', the intersection of c'P with the perpendicular cc' to the ground line. The horizontal trace of the plane containing sP and AB is the line AB itself, since it is in the ground plane. The vertical trace passes through c', the trace of the line sc, Pc', which is contained in the plane; it must also pass through A, therefore the vertical trace is the line Ac'. Hence Ac' is the perspective of AB.

48. PERSPECTIVE OF A POINT NOT IN THE GROUND PLANE.

The construction given in § 45 does not change, when the point to be placed in perspective is not in the ground plane. A line is still drawn through the station sP, Fig. 65 and the point mm'. The vertical trace μ is the perspective of mm'.

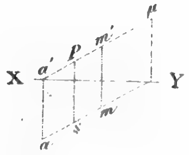

Fig. 65

The horizontal trace a of the visual ray is the perspective of the point mm' on the ground plane; hence it may be stated as a general rule, that the perspectives of a point on the ground and picture planes are the traces of the line joining the station to the point

49. PERSPECTIVE OF A LINE NOT IN THE GROUND PLANE.

Let $ab, a'b'$, Fig 66, be a line not in the ground plane: to obtain its perspective a plane must be passed through the station sP and the line $ab, a'b'$; the intersection of this plane with the picture plane that is the vertical trace of the plane, is the perspective of the line.

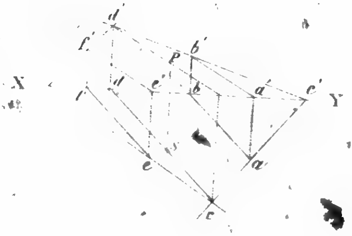

Fig.66

Through sP, draw a parallel sd, Pd' to $ab, a'b'$; both lines are contained in the plane to be drawn, therefore the traces of the plane are the lines $ac, d'b'$ joining the traces of same denomination of the parallels; and $d'b'$ the vertical trace of the plane, is the perspective of $ab, a'b'$.

Let us now consider another line, $ef, e'f'$ parallel to $ab, a'b'$; the plane passing through $ef, e'f'$ and the station, sP, must again contain the parallel sd, Pd', through the station; therefore the vertical trace of the plane, which is the perspective of $ef, e'f'$ is the line $f'd'$ joining the vertical traces of the two parallels. Hence the perspective of any line parallel to $ab, a'b'$ will pass through the point d'. This result could be foreseen, because when a system of parallels has to be placed in perspective, all the planes serving to project them on the picture plane have a common line of intersection, parallel to the general direction of the system and passing through the station. Its trace on the picture plane must therefore be the common point of intersection of the perspectives. This point is called the "Vanishing point" of the parallel lines, because it represents the parts of the lines which are at infinity; the perspective ends or vanishes at that point.

The horizontal traces of the planes are the perspectives of the parallel lines on the ground plane. Like the perspectives of the picture plane. they all meet in a common point, which is the horizontal trace of the parallel line through the station: it is the vanishing point of the perspectives of the ground plane. Therefore it is seen that when a plane is drawn through the station and a line in space, the traces of the plane on the picture and ground planes are the perspectives of the line on those planes.

50. POSITIONS OF THE VANISHING POINT. A horizontal line has its vanishing point on the horizon line

because the parallel drawn through the station, being horizontal, is all contained in the horizon plane and has its vertical trace on the horizon line.

Perpendiculars to the picture plane being parallel to the distance line have for vanishing point the vertical trace of the distance line which is the principal point of the perspective.

The vanishing points of horizontal lines making an angle of 45° with the distance line are called "distance points" (D,D Fig.67): their distance from the principal point is equal to the distance line, because a horizontal line inclined at 45° with SP, forms, an isosceles triangle SPD in which SP = PD.

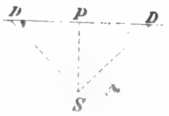

Fig.67

Lines in the principal plane have their vanishing point on the principal line. Two of these lines form angles of 45° with the distance line, one above, and the other below the horizon. Their vanishing points are known as "upper and lower distance points"; they are also at the same distance from the principal point as the station.

Lines parallel to the picture plane have no vanishing point. It will be shown later on that their perspectives are parallel to the lines themselves and do not meet.

51. VANISHING LINE. Through the station SP, Fig. 68, pass a plane TVZ parallel to a given plane QRM. The vertical trace VZ contains the traces of all the lines drawn through the station parallel to QRM, it is therefore the locus of the vanishing points of parallels to the plane QRM. This trace VZ will be called the "vanishing line" of the plane QRM or of any other plane parallel to it (1).

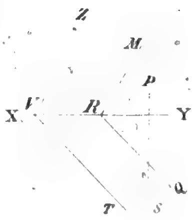

Fig. 68

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(1) The term "vanishing line" is usually applied to the perspectives of parallel lines: admitting that the expression "vanishing point" is a proper one, the line VZ cannot be called otherwise than "vanishing line". The term is used here with that acceptation only.

The horizontal trace VT is in like manner the vanishing line of the perspectives on the ground plane.

52. LINES OR FIGURES IN FRONT PLANES. The perspective of a straight line contained in a front plane is another straight line parallel to the first one. For the plane containing the station and the given line being cut by two parallel planes, the picture and front planes, the intersections are parallel lines. But these intersections are the line itself and its perspective, therefore the perspective is parallel to the given line.

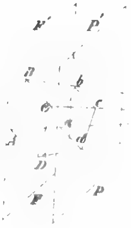


Fig. 69

Let S, Fig. 69 be the station, PP', FF', the picture and front planes and ABCD a polygon in the front plane. Join SA, SB, SC, SD; these lines intersect the picture plane at *a, b, c, d,* the polygon *abcd* being the perspective of ABCD. The lines drawn from S form a pyramid of which the polygon of the front plane is the base and the perspective a section by a plane parallel to the base. It is shown in Geometry that when a pyramid is cut by a plane parallel to the base, the section is a figure similar to the base. The front plane being parallel to the picture plane, the perspective must be similar to the original figure.

It follows that a curve in the front plane is represented by a similar curve in perspective, because such a line can be assimilated to a polygon with a great number of sides.

When the front plane is beyond the picture plane, as in Fig.69, the perspective is smaller than the original figure; it is larger when the front plane is between the station and the picture plane, but in either case it is an exact representation of the figure itself, on a different scale. This scale, or the proportion between the perspective and the original figure, is called the "scale of the front plane" it is the proportion of the distance line to the distance between the station and the front plane.

A straight line parallel to the picture plane is contained in a front plane and is represented in perspective by a line parallel to itself, therefore parallel lines which are also parallel to the picture plane have parallel lines for perspectives and have no vanishing point. The parallel to the given lines passing through the station, being parallel to the picture plane has no trace on it.

Vertical lines are parallel to the picture plane and appear in perspective as parallels to the principal line.

Horizontal lines parallel to the picture plane are in perspective parallel to the horizon line.

53. MEASURING LINES AND MEASURING POINTS. Let PP, Fig. 70, be the picture plane, S the station and AB a straight line piercing the picture plane at A. Through S, draw the parallel SV to AB: V is the vanishing point of AB whose perspective is VA, since the vertical trace A is a point of the perspective and the vanishing

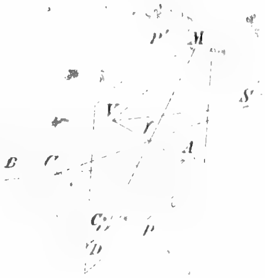


Fig. 70

point is another one.

Through V, draw VM equal to VS and through A the line AD parallel to VM. Take a point C of AB and join CS, the intersection γ with VA is the perspective of C. Join $M\gamma$ and produce to its intersection C_1 with AD.

VS and AB being parallel to each other, the triangles $V\gamma S$ and $A\gamma C$ give the proportion:

$$\frac{VS}{AC} = \frac{V\gamma}{A\gamma} \qquad (1)$$

The triangles $VM\gamma$ and $AC_1\gamma$ are also similar, VM being parallel to AC_1, therefore:

$$\frac{V\gamma}{A\gamma} = \frac{VM}{AC_1} \qquad (2)$$

Hence from (1) and (2):

$$\frac{VS}{AC} = \frac{VM}{AC_1}$$

But by construction

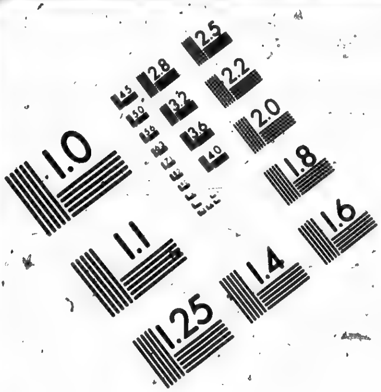
$$VM = VS$$

therefore

$$AC = AC_1$$

The line AC, represented in perspective at $A\gamma$, is equal to the line AC_1.

Fig. 71 shows the picture plane with the same



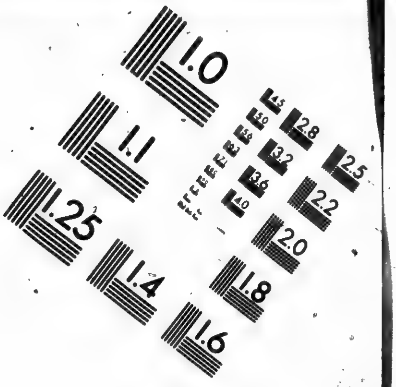
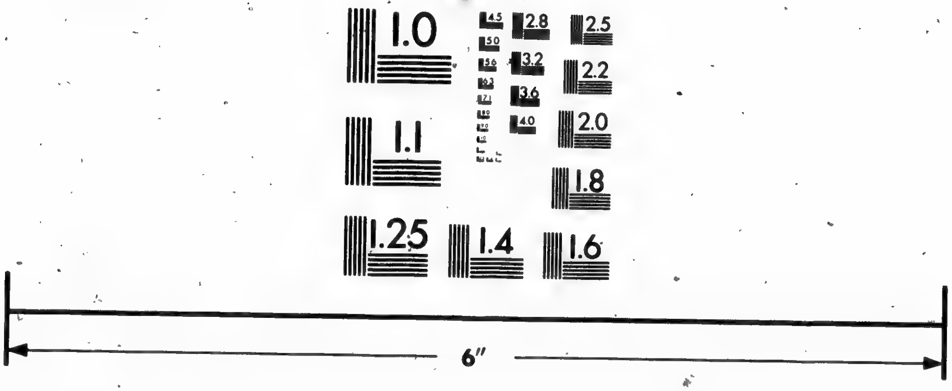
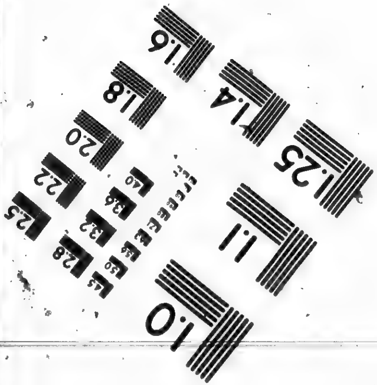
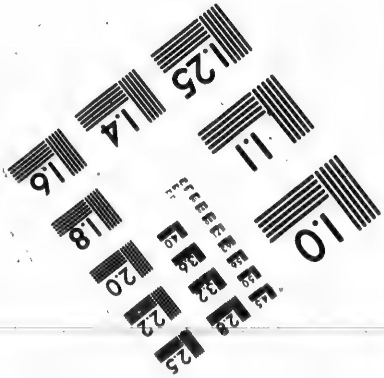


IMAGE EVALUATION TEST TARGET (MT-3)







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letters as Fig.70. The part of the line seen in perspective at $A\gamma$ is equal to AC_1. On AC_1 take another point D, join to M, and call δ the intersection with VA. The line seen in $A\delta$ is equal to AD, therefore the part seen in $\gamma\delta$ is equal to C_1D.

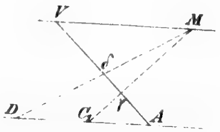


Fig.71

The line AD is called the "measuring line" of AV, because it serves to measure the length of the line in space corresponding to any portion of its perspective AV; M is the "measuring point".

VM was not drawn in any particular direction, therefore the direction of the measuring line, parallel to VM, is indeterminate. It is usual to make it parallel to the horizon line.

The position of the measuring point depends only on the vanishing point, therefore the same measuring point may serve for all lines parallel to the same direction.

The same measuring line will serve for all lines having their vertical traces on it. Should the line VM be drawn parallel to the vertical trace of a plane, this trace would be a measuring line for all lines contained in the plane.

If the measuring line is taken parallel to the horizon, the measuring point of any horizontal line is on the horizon line, since the vanishing point is on that line. All lines in the same horizontal plane have then for measuring line the vertical trace of the plane, and lines in the ground plane have the ground line.

There is no measuring line or point for lines in a front plane, because they have no vertical trace or vanishing point; the scale of the front plane has to be employed when the length of such a line is wanted.

The distance points are measuring points for lines parallel to the distance line.

54. REDUCTION OF A PERSPECTIVE TO SCALE. Hitherto, it has been assumed that in the constructions, the real dimensions of the figures were employed. It would be quite impracticable to do so in the

generality of cases; the dimensions must be reduced to a certain scale in order not to exceed the limits of the paper.

A change in the position of the measuring line permits the use of reduced distances. Let V, M, and AC, Fig. 72, be the vanishing and measuring points and the measuring line of the perspective AV. The part of the line seen in $\beta\gamma$ is equal to BC. Through a point a of AV, draw the parallel ac to AC and let us use it as a measuring line: the length corresponding to $\beta\gamma$ is bc and we have the proportion:

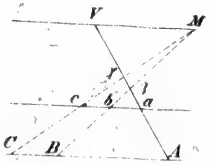


Fig. 72

$$\frac{cb}{CB} = \frac{Va}{VA}$$

Thus the lengths obtained are all reduced in the proportion of $\frac{Va}{VA}$. Therefore in order to obtain at once the length, on a certain scale, of a line seen in perspective, it is sufficient to reduce the distance between the measuring line and the vanishing point in the proportion of the scale to be employed. Thus if Va be made the one thousandth part of VA, the distances will be obtained on a scale of $\frac{1}{1000}$. M is the measuring point and ac the measuring line, of a line having V for vanishing point and a for trace on the picture plane; the new line is therefore parallel to the line joining V to the station and to the original line seen in perspective, but its distance from the station has been reduced to the scale adopted.

Hence, to obtain the length reduced to scale of a line seen in perspective, reduce to scale the distance of the line from the station, moving it parallel to itself in the plane containing the station.

The same conclusion is arrived at in a more direct manner otherwise. A figure ABCD, Fig. 73, forms with the visual rays joining it to the station, a pyramid, the intersection of which by the picture plane is the perspective $\alpha\beta\gamma\delta$. Let the pyramid be

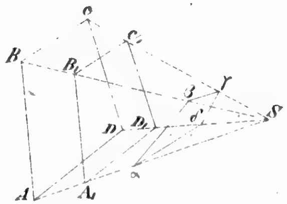


Fig. 73

cut by a plane parallel to the base ABCD: the intersection $A_1 B_1 C_1 D_1$ is similar to ABCD, the proportion being $\frac{SA_1}{SA}$. The lines $A_1 B_1$, $B_1 C_1$ ......measured by means of their perspectives $\alpha\beta$, $\beta\gamma$ ...... will therefore be the lines AB, BC,...... reduced to the scale $\frac{SA_1}{SA}$ The same demonstration applies to any system of figures, whenever every point of the system has been moved in a straight line towards the station, so as to reduce its distance from the station in the proportion of the scale given. Hence we deduce the following important rule:-

To lay off dimensions reduced to scale or to measure them from a perspective, assume that the system formed by the station and the original figures or objects, had been reduced to scale when the perspective was executed.

55. TO PLACE IN PERSPECTIVE A POINT OF THE GROUND PLANE.

1st. By means of the principal point and a distance point.

Let M, Fig. 74, be the point, XY the ground line, P and D the principal and distance points, the picture plane being revolved upon the horizontal plane. Draw MA at an angle of 45° and MB perpendicular to the ground line. The perspective of AM is the line AD joining the trace on the picture plane to the distance point. The perspective of MB vanishing at the principal point is PB, therefore the perspective of M is at μ.

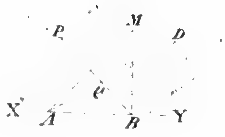


Fig. 74

2nd. By means of the distance of the point from the ground line.

Draw MB perpendicular to XY and take AB equal to MB. Join AD and PB.

3rd. By means of the station and principal point.

Join the foot of the station *s*, Fig. 75, to the point M. The line *s*M is the horizontal trace of the vertical plane containing M and the station, which plane cuts the picture plane on a line AC perpendicular to XY. From M draw the perpendicular MB to the ground line; it is represented in perspective by PB, therefore μ the intersection of AC and PB is the perspective of M.

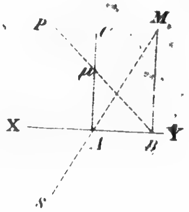


Fig. 75

4th. By means of the projection on the principal plane.

Revolve the principal plane around its trace sp, Fig. 76, upon the ground plane: the station will come in S_1 on a perpendicular to sp, sS_1 being equal to the altitude of the station. Draw Mm' perpendicular to sm' and join S_1m': it is the projection on the principal plane of the visual ray from the station to the point M, and its intersection μ_1 with pY is the projection of the perspective of M revolved upon the ground plane. Join sM and at m erect a perpendicular $m\mu$ to the ground line: the perspective of M is on that perpendicular at a distance $m\mu$ equal to $p\mu_1$.

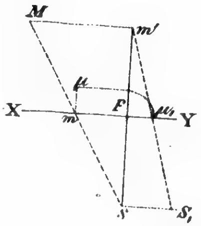


Fig. 76

When a great number of points have to be placed in perspective, this last method is very convenient. In practice the perspective is not constructed on the ground plan itself, as the operations would become confused: the plan and perspective are kept separate.

Let ABCD, Fig. 77, be the ground plan, $X_2 Y_2$ the ground line, and $s_2 p_2$ the trace of the principal plane. Join s_2 to A, B, C, and D.

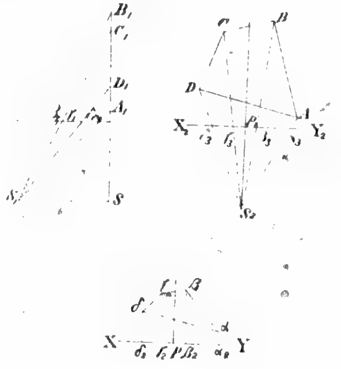


Fig. 77

On the paper which is to serve for the perspective, draw the ground line XY and take a point p as intersection of the principal plane. Take on the edge of a piece of paper the distances from p_2 to α_3, β_3, γ_3, δ_3, and carry them on XY in α_2, β_2, γ_2, δ_2; At the last mentioned points erect perpendiculars to the ground line.

At another place draw a line $s_1 B_1$ to represent the intersection of the ground and principal planes:

place the station S_1 at its height h above the ground plane, take s_1p_1 equal to the distance line and draw the trace of the picture plane, $p_1\beta_1$ perpendicular to s_1B_1.

On the edge of a piece of paper, take the distances of A, B, C, D, from the ground line X_2 Y_2 and carry them on p_1B_1. Join S_1 to A_1, B_1, C_1, D_1. Again take on the edge of a piece of paper the distances of α_1, β_1, γ_1, δ_1, from p_1 and lay them on the perpendiculars $\alpha_2\alpha$, $\beta_2\beta$, $\gamma_2\gamma$, $\delta_2\delta$. This gives $\alpha\beta\gamma\delta$ as the perspective of ABCD.

56. **TO PLACE IN PERSPECTIVE A LINE OR FIGURE OF THE GROUND PLANE.** A line of the ground plane may be placed in perspective by determining the perspectives of two of its points.

When the vanishing point is known, only one additional point is required to define the perspective.

With a figure composed of straight lines, the perspectives of the points of intersection are fixed and joined together by straight lines.

The perspective of a curve is found from the perspectives of a sufficient number of points or by tangents to the curve.

57. TO PLACE IN PERSPECTIVE A POINT OUTSIDE OF THE GROUND PLANE. When a point is not in the ground plane, the perspective of its horizontal projection is first found: the height of the point above or below the ground plane is next reduced to the scale of the front plane and laid on the vertical of the perspective previously found.

Let m, Fig. 78, be the projection of the point on the ground plane and B the perspective of m, obtained as in § 55. From m let fall the perpendicular mA on XY and take Am' equal to the height of the point. Join m' to the principal point, P; Pm' is the perspective of a perpendicular to the picture plane from M, therefore the perspective of M must be on Pm'. But the given point M is on the vertical line passing through m whose perspective is the perpendicular $B\mu$ to the ground line, therefore the perspective of the point M is at the intersection

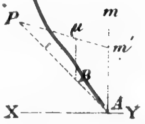


Fig. 78

μ of the two lines.

Comparing Fig.78 to Fig.65,§ 48,it will be seen that the construction is precisely the same,although made on different principles.

58. TO PLACE IN PERSPECTIVE A LINE OUTSIDE OF THE GROUND PLANE. When a line is in a horizontal plane, that plane may be taken temporarily as ground plane and changed when the perspective has been obtained.

If in any other plane,the perspective may be found by means of the vanishing point and horizontal trace. The latter is placed in perspective as explained in § 55.and joined to the vanishing point.

For lines in front planes,one point of the line is placed in perspective and through it,a parallel to the line is drawn.

59. THE DISTANCE LINE IS AN AXIS OF SYMMETRY OF THE PERSPECTIVE. A perspective is symmetrical with reference to the distance line,all points of the picture plane at the same distance from the principal point having the same geometrical properties. Therefore any plane perpendicular to the picture plane may be taken as ground plane,or any line through the

principal point as horizon line. So when figures are contained in a plane perpendicular to the picture plane their perspectives can be obtained by taking the plane of the figures for ground plane and its vertical trace for ground line.

60. GIVEN THE HEIGHTS OF TWO POINTS AND THEIR PERSPECTIVES, TO FIND THE VANISHING POINT AND TRACE ON PICTURE PLANE OF THE LINE JOINING THE GIVEN POINTS.

Let HH′ and P, Fig. 79, be the horizon line and principal point. α and β two points of the perspective. Draw EF parallel to the horizon line at a distance equal to the height of α : it is the trace of the horizontal plane containing the point of space corresponding to α. The perspective of the perpendicular to the picture plane passing through α is Pα : its vertical trace is C. Draw CD perpendicular to the horizon line and equal to the height of β above the plane of α, D is a point of the picture plane at the same height as β, and PD is the

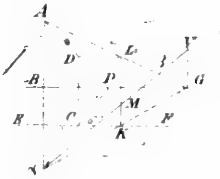


Fig. 79

perspective of the perpendicular to the picture plane passing through D; PD is in the same vertical plane as PC and if produced will meet the vertical of α seen in perspective at αB. The point of intersection A is at the same height as D and β, therefore $A\beta$ is a horizontal line and its vanishing point is on the horizon line at G. But $A\beta$ and $\alpha\beta$ are in the same vertical plane having for vertical trace the perpendicular GV to the horizon line, therefore the vanishing point of $\alpha\beta$ is at its intersection V with GV.

To find the trace, draw through D the parallel DL to the horizon line: it is the trace on the picture plane of the horizontal plane containing AG, and the trace of AG is at its intersection L with DL. But AG and αV being in the same vertical plane, the trace of αV is in M, on the perpendicular LM to the horizon line.

αG is a horizontal line also in the same vertical plane as AG and αV: consequently its trace is K, on LM produced. But αG is in the horizontal plane whose trace is EF, therefore K is the intersection of αG and EF.

61. TO FIND THE INTERSECTIONS OF A VERTICAL LINE BY A SERIES OF HORIZONTAL PLANES Let HH' and FG, Fig.80, be the horizon and principal lines of a perspective, μ a point of the perspective $\eta\theta$ of a vertical line, of which the altitude is known. Take PM equal to this altitude: M will be the point of the picture plane having the same altitude as μ. Join μM: it is the perspective of a horizontal line and its vanishing point is V. Mark on FG the intersections A, B, C, D, E of the horizontal planes, join to V and produce VA, VB, VC, VD, VE to $\eta\theta$; these lines are the perspectives of horizontal lines parallel to μ M and contained in the horizontal planes. Their intersections $\alpha, \beta, \gamma, \delta, \varepsilon$, with the perspective of the vertical are the points required.

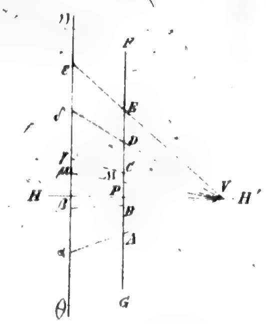


Fig.80

This construction is employed for determining the intersections of a vertical line by contour planes: the equidistance is marked on the edge of a piece of paper which is pinned along GF so that P corresponds to the altitude of the station. A straight edge is placed on μ and the point of same height of the equidistance scale, then a pin is planted at V and the straight edge moved through each of the points A B..... E. always keeping it in contact with the pin.

Another solution consists in projecting the vertical line and its perspective on the principal plane

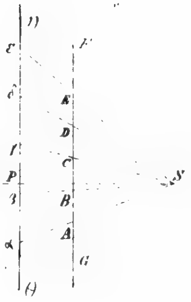


Fig. 81

Let SP, Fig 81 be the distance line, $\eta\theta$ the principal line and FG the intersection of the front plane containing the vertical line, by the principal plane. Mark on FG the intersections of the horizontal planes; join to S and produce to $\eta\theta$ the intersections are the projections on the principal plane of the points required.

In practice, the construction is made on the perspective: $\eta\theta$ Fig. 82, being the perspective of the vertical line, NM is taken on the horizon line equal to the distance line and NQ equal to the distance of the vertical line from the picture plane. At Q a perpendicular is erected to HH' and the equidistance scale pinned alongside, so that Q shall correspond to the altitude of the station. The construction is completed as in Fig.81.

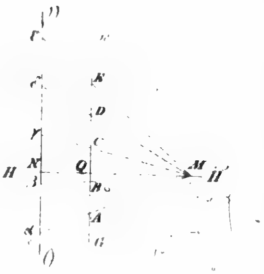


Fig.82

Contour planes being equidistant, the divisions $\alpha\beta$, $\beta\gamma$......... of the perspective are equal: it would therefore be sufficient to find the length of one division and to carry it on the perspective of the vertical line.

62. **TO MARK ON THE PERSPECTIVE OF ANY LINE OR CURVE CONTAINED IN A VERTICAL PLANE, THE INTERSECTIONS BY A SERIES OF HORIZONTAL PLANES.** Let $\mu\delta$, Fig.83, be the perspective of a line contained in a vertical plane: that plane contains the vertical seen in perspective at δM, perpendicular to the horizon line. Mark the points of division A,B,C of δM by the

horizontal planes, (§ 61) and join μ to the point of the perspective δM of same altitude. This being the perspective of a horizontal line, its vanishing point is V. Join V to A, B, C: these lines are parallel to $M\mu$, therefore they are in the plane $\mu M\delta$ and will intersect the line seen in perspective at $\mu\delta$, but they are also contained in the horizontal planes, hence α, β, γ, are the points required.

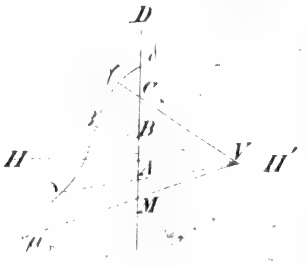


Fig 83

Instead of dividing first the vertical line δM the trace on the picture plane and vanishing point of μM may be determined as in § 60 and the points of intersection marked at once on the line $\mu\delta$ by placing the equidistance scale on the perpendicular to the horizon line passing through the vertical trace and joining the points of division to the vanishing point.

When the horizontal projection of the line is known, the vanishing point and trace can be obtained at once. Let *α*, *β*, Fig.84, be the perspective of the line, *ab* its horizontal projection, HH′ the horizon and XY the ground line. The intersection of the ground plane by the vertical plane containing the line seen at *αβ* is *ab*; the trace on the picture plane of this intersection is at E, where *ab* produced meets XY.

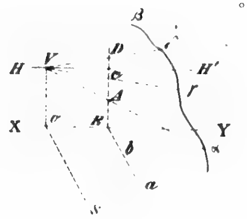


Fig.84

Through the foot of the station *s*, draw *sv* parallel to *ab* and *v*V perpendicular to XY meeting the horizon line in V. V is the vanishing point of parallels to *ab*. But the intersections of the horizontal planes by the plane of *αβ* being parallel to *ab*, V is their vanishing point; and since they are all in the same vertical plane, their traces are on the vertical ED of the picture plane. Hence the equidistance scale is to be placed along ED, taking care that the point E of the scale corresponds to the altitude of the ground plane; the divisions of the scale are joined to the vanishing point and produced to their intersection with the perspective.

63. TO MARK ON THE PERSPECTIVE THE INTERSECTIONS OF A PLANE, LINE OR CURVE BY A SERIES OF HORIZONTAL PLANES.

The intersections of a plane by a series of horizontal planes are horizontal lines parallel to the trace on the ground plane of the plane intersected; the vanishing point of these lines is the point of intersection of the horizon line by a parallel to this trace, drawn through the station.

Let $\alpha\gamma$, Fig.35, be the perspective of a line or curve in the plane POP', XY the ground line and HH' the horizon line. Through the foot of the station s draw sv parallel to OP and erect the perpendicular vV to the ground line meeting the horizon line in V: V is the vanishing point of horizontal lines in the plane POP', and consequently, of the intersections of that plane by the horizontal planes. The traces of these

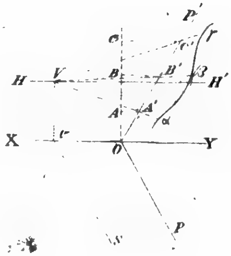


Fig.85

lines on the picture plane are on OP′ and the vertical distance between them is that of the horizontal planes:therefore place at O,on a perpendicular to the ground line,the distances of the horizontal planes or the scale of equidistance,draw parallels to the ground line through the divisions A,B,C of the scale and join A′,B′,C′ to the vanishing point. These lines are the perspectives of the intersections of the plan POP′ by the horizontal planes.

64. INTERSECTIONS OF A PRISM,PYRAMID OR CONIC SURFACE BY A SERIES OF HORIZONTAL PLANES. The intersections of a prism or pyramid by a series of horizontal planes can be drawn on the perspective by determining the intersections of the edges of the prism or pyramid by the planes and joining the corresponding points by straight lines

A similar process can be applied to a conic surface by using generatrices instead of edges,and also by employing tangents to the intersections,parallel to the tangents drawn to the curve of the ground plane forming the base of the cone.

65. TO PLACE A POINT OF THE GROUND PLANE BY MEANS OF

ITS PERSPECTIVE. To restore a figure by means of its perspective is the converse of perspective. Let us consider first the case of a point of the ground plane: its place can be found by inverting any of the constructions given in § 55.

For instance, in Fig. 74, the perspective μ of the point is joined to the principal and distance points, P and D. At B a perpendicular BM is erected to the ground line and at A a line AM is drawn at an angle of 45° with the ground line. M is the point of the ground plane.

In Fig. 75, join $P\mu$ and draw μA and BM perpendicular to the ground line. Join sA and produce to the intersection with BM.

In Fig. 76, $p\mu$ is taken equal to the distance μm of the perspective μ from the ground line and μ_1 is joined to the station S_1 revolved on the ground plane. The foot of the station, s, is joined to the foot of the perpendicular μm to the ground line and the point M is at the intersection of sm produced, by the parallel to the ground line through m'.

When a number of points have to be placed, the constructions are made as in Fig.77, but in inverse order. The perspective $\alpha\beta\gamma\delta$ is given: the distances $p\alpha_2, p\beta_2$, ... are carried on $X_2 Y_2$; $\alpha\alpha_2, \beta\beta_2$ ..... on $p_1\gamma_1$. Then s_2 is joined to α_3, β_1 ......and on these lines produced the points A, B, C, D, are so placed that their distances from the ground line are equal to

p_1A_1, p_1B_1, p_1C_1, p_1D_1.

66. TO PLACE A LINE ON THE GROUND PLANE BY MEANS OF ITS PERSPECTIVE The trace of a line on the picture plane is the point where its perspective intersects the ground line: the point is common to the perspective and to the line (β, Fig.86)

The vanishing point, V, is the intersection of the perspective by the horizon line; it gives the direction of the line of the ground plane.

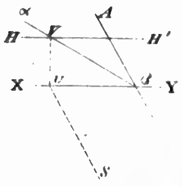


Fig.86

From V, let fall the perpendicular Vv to XY and join vs. Through β, draw βA parallel to sv; it is required line.

With a front line a point of the line is fixed by one of the methods of § 65 and a parallel to the ground line drawn through the point.

67. TO DRAW A FIGURE ON THE GROUND PLANE BY MEANS OF ITS PERSPECTIVE. A figure of the ground plane may be constructed by means of its perspective as described in § 65, each of the summits of the figure being determined separately.

It may also be constructed by determining each of the lines forming the figure, as in § 66.

An irregular figure is enclosed between straight lines and drawn at sight.

A convenient method is that known as the "method of squares". The ground plane is divided into squares by lines parallel and perpendicular to the ground line; the network of squares is projected on the perspective and the figure drawn at sight in the corresponding squares.

To construct the perspective of the squares, the distances of the parallel lines are marked on the ground line in A, B, C, D, E, Fig. 87, the perspectives of the perpendiculars to the ground line are obtained by joining these points to the principal point P.

The principal plane is next plotted separately, SK_1 being its trace on the ground plane, S the station and P_1P' the trace of the picture plane. Mark the intersections F_1, G_1, H_1, K_1 of SK_1 by the lines parallel to the ground line, join to S and carry to Pp the distances of p_1 from F_2, G_2, H_2, K_2: through the points so obtained, F, G, H, K, draw parallels to the ground line, which will complete the perspective of the squares.

It is not necessary that the sides of the squares be parallel or perpendicular to the ground line. Any other direction may be adopted, as, for instance, north and south, and east and west in the case

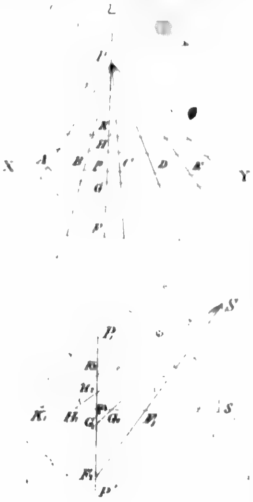
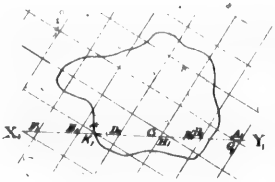
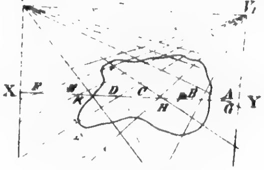


Fig.87



Ground Plan



Perspective

Fig.88

of topographical perspectives

The vanishing points V and $V_,$,Fig.88,of these lines are found as usual by drawing through the station parallels to their directions,to the intersection with the horizon line.

The points of intersection with the ground line of the north and south lines,which will be supposed to vanish at V,are taken from the ground plan,carried to the ground line of the perspective,in A,B,C,D,E,F, and joined to V:this gives the perspective of one set of parallel lines. The other set is obtained by a similar process,carrying the points $G_,$,$H_,$,$K_,$,$L_,$, from the ground plan to the perspective and joining to the vanishing point $V_,$.

The squares must be made small enough to guide the draughtsman accurately in transferring the figure from the perspective to the ground plan.

68. VANISHING SCALE. The direction of a point of the ground plane is easy to find:it is sufficient to join the foot of the station to the projection of the perspective on the ground line. Were the distance of the point determined,it could be located at once.

This is done by means of the "vanishing scale".

Fig.89, represents the principal plane; Pp and pA are the traces of the picture and ground planes and S the station at a height h above the ground plane. On pA and on each side of p, mark equal distances, 100, 200, etc: they represent the intersections of pA by parallels to the ground line. Join these points to S: the perspectives of the above parallels will be parallels to the ground line passing through the points of division of pP. Suppose now that the distance of a point of the perspective from the ground line be found equal to pm: then the point of the perspective is on a parallel to the ground line passing through m. But this line is the perspective of a parallel to the ground line passing through M, therefore the point to be found, being on that parallel, is at the distance pM from the ground line. M and m corresponding to the same divisions of the scales

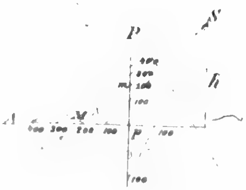


Fig.89

*p*P and *p*A, the distance of the point is obtained at once by reading the division of *p*A corresponding to *pm*.

The scale constructed as above on *p*P is called a vanishing scale; when the distance line is constant, the scale is the same for all planes at the same altitude below the station.

69. USE OF THE MEASURING LINE. Sometimes the greater part of an irregular figure may be enclosed between two parallel lines, as in Fig.90. A point V is taken on the horizon line such that two lines drawn from it will enclose the figure $\alpha\beta\gamma\delta\varepsilon$ as well as possible. These lines are the perspectives of two parallel lines in the ground plane and their vanishing point is V. Draw these parallels on the ground plan in $F_1 E_1$ and $G_1 D_1$ and place on the perspective the measuring point by taking VM equal to the distance of V from the station.

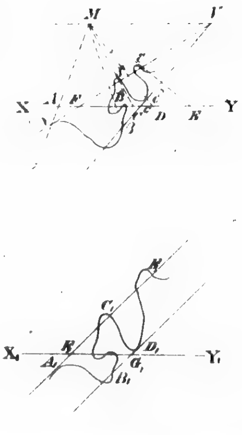


Fig.90

The measuring line is the ground line XY. Find the distances from F and G to the points of the parallels corresponding to various points of the irregular figure and transfer them in $A_{,}$, $B_{,}$, $C_{,}$, $D_{,}$, $E_{,}$, to the ground plan. Draw the intermediate parts of the figure at sight.

Should two parallel lines prove insufficient, the number can be increased.

The method of squares, the vanishing scale and the measuring line can be employed for finding the perspective from the ground plan. The operations are the converse of the preceding ones and require no further explanation.

70. PRECISION OF THE METHOD. Let Ss and M, Fig. 91, represent the vertical plane passing through the station and a point of the ground plane, Sm and μA, the traces of the horizon and picture planes and μ the perspective of M. Draw Mm perpendicular to Sm: it is the height, h, of the station above the ground plane.

The similar triangles $SA\mu$ and SmM give:

$$\frac{Sm}{mM} = \frac{SA}{A\mu}$$

h

or

$$\frac{y}{h} = \frac{l}{x} \qquad (1)$$

To find the effect on the distance y from the station to M of an error dx in the perspective, the equation (1) must be differentiated, considering x and y as variables; this gives:

$$dy = -\frac{y}{x}dx = -\frac{y^2}{hl}dx \qquad (2)$$

So the error in the position of M caused by an error in the perspective increases as the square of the distance: therefore the method must not be employed for points or figures at too great a distance from the station.

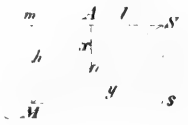


Fig. 91

The error decreases as the height of the station increases: thus if the height be doubled, the error will be reduced to one half. Hence perspectives intended for the reproduction of figures in the ground plane should be taken from as great a height as possible.

The error decreases also as l increases, or as the size of the perspective increases.

71. TO DETERMINE FROM THE PERSPECTIVE, THE PROJECTIONS OF A POINT NOT IN THE GROUND PLANE, BUT OF WHICH THE HEIGHT IS KNOWN. The perspective of a point is not sufficient to determine its position: other data must be furnished, such as the traces of a plane containing it, its distance, or its height above the ground plane.

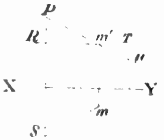


Fig. 92

If the height be known, draw a parallel RT, Fig. 92, to the ground line representing the trace on the picture plane of the horizontal plane containing the point. The projections of the visual ray joining the station to the point are sm, $P\mu$ (§ 47): it pierces the horizontal plane RT in m, m', and as the point to be found is in that plane and on the line sm, $P\mu$, it is their point of intersection, mm'.

The construction is not always possible. For instance RT may pass through P: this means that the point is in the horizon plane, in which case it cannot be located by means of its perspective.

$P\mu$ may coincide, or very nearly, with Ps, and the construction become impossible or uncertain. The visual ray joining the station to the point is then projected on the principal and ground planes instead of of the picture and ground planes: the different steps are precisely the same in both methods.

72. TO CONSTRUCT FROM ITS PERSPECTIVE A FIGURE IN ANY HORIZONTAL PLANE. The methods given in § 65,66, and 67 apply to figures in any horizontal plane, by using the planes of the figures as ground planes; all that is required being to shift the ground line on the perspective to its proper position.

73. TO FIND THE TRACES AND VANISHING POINT OF A LINE GIVEN BY ITS HORIZONTAL PROJECTION AND PERSPECTIVE. Before proceeding to consider figures in various planes, it is necessary to show how the plane of a figure and the traces of straight lines can be determined.

Let $\alpha\beta$ and ab, Fig.93, represent the perspective and horizontal projection of a line. At b erect a perpendicular to the ground line: the trace of the line on the picture plane must be on that perpendicular and also on $\alpha\beta$, therefore it is at their intersection β.

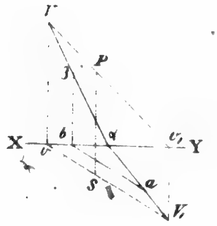


Fig. 93

The vanishing point is the trace of a parallel to the line, drawn through the station: the horizontal projection of this parallel is sv, drawn through the foot of the station parallel to ab and its trace is on the perpendicular vV to the ground line. But this trace is the vanishing point of $\alpha\beta$, therefore it is at the intersection of vV and $\alpha\beta$ produced.

The vertical projection passes through the trace V and the principal point P; producing it to the intersection with XY and erecting the perpendicular $v_{,}V_{,}$ to XY, the trace on the ground plane is found at $V_{,}$.

The line joining $V_{,}$ to α is the perspective on the ground plane of the given line (§ 48) whose trace is the intersection of $\alpha V_{,}$ and ab.

The trace on the ground plane may also be found by revolving the projecting plane of ab around its vertical trace $b\beta$, Fig.94, upon the picture plane.

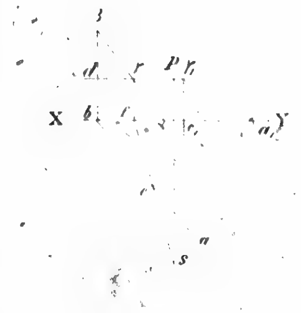


Fig.94

Draw the horizon line Pd : the trace of the given line on the horizon plane is seen in γ on the perspective: its horizontal projection is at the intersection c of ab by the line joining the foot ofthe station to the foot f of the perpendicular γf to XY. When the projecting plane revolves, c describes the arc of circle cc_1 with b as a center: the point of the given line corresponding to γ moves in the horizon plane, therefore it will come in γ_1 on the horizon line at the intersection with the perpendicular $c_1\gamma_1$ to XY. The revolved line is βa_1 and the revolved trace on the ground plane is a_1 : Revolving a_1 back to ab, the trace is obtained in a.

The angle formed in a_1 by the revolved line and XY is the angle of the line with the ground plane.

A third method consists in determining from the perspective the heights of two points of the given line, as will be explained later on.

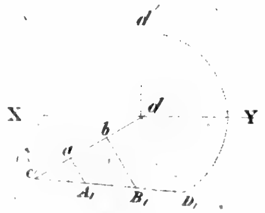


Fig. 95

The projecting plane of the line is revolved on the ground plane around the horizontal projection ab, Fig. 95. The points A and B fall in $A_{,}$ and $B_{,}$, the perpendiculars $aA_{,}$ and $bB_{,}$ to cd being the heights of A and B above the ground plane. $A_{,}B_{,}$ is the revolved line and c its trace on the ground plane. The revolved trace on the picture plane is at the intersection of $A_{,}B_{,}$ produced with the perpendicular $dD_{,}$ to cd: it is revolved back to the picture plane by erecting a perpendicular dd' to XY and describing an arc of circle with d as center and $dD_{,}$ as radius.

74. GIVEN THE SLOPE OF A LINE AND THE HORIZONTAL PROJECTION OF ONE OF ITS POINTS, TO FIND THE HORIZONTAL PROJECTION AND TRACES OF THE LINE. Let a, Fig. 96, be the horizontal projection of a point of the

line seen in perspective in $\alpha\beta$, s the foot of the station and XY the ground line. Join sa, produce to m and erect the perpendicular $m\alpha$ to XY: α and a are the perspective and projection of the same point, A, of the given line. Draw the horizon line HH′: δ is the perspective of the trace of the given line on the horizon plane. Revolve the projecting plane of the line around the vertical of a until parallel to the vertical plane: the point A of the given line, being on the vertical of a, does not move, and its perspective remains in α. The perspective δ of the trace on the horizon plane moves on the horizon line: when the projecting plane is parallel to the vertical plane, the perspective of the revolved line is parallel to the line itself and may be drawn in $\alpha\delta_1$, since the angle $\alpha\delta_1 H$ is given. The trace of the projecting plane on the ground plane

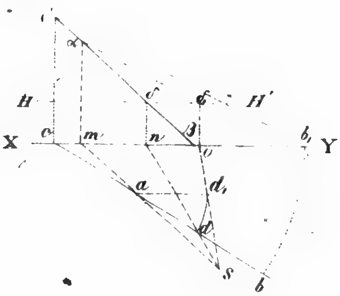


Fig. 96

has come in ad_1 parallel to the ground line. The point d_1 of the horizontal projection corresponding to δ_1 of the revolved perspective is obtained by letting fall $\delta_1 O$ perpendicular to XY and joining sO.

Revolving back the projecting plane to its original position, δ_1 comes in δ and the corresponding point of the horizontal projection must be on the line sn_1 joining the foot of the station to the intersection n with the ground line of a perpendicular from δ. But this corresponding point is the new position of d_1, and d_1 moves on an arc of circle with a as center, therefore d_1 comes in d and da is the horizontal projection of the given line.

The vertical trace is found at c' by the usual construction: the vertical projection and horizontal trace may be determined as in § 73 or the triangle formed by cc', cb and the given line may be revolved around cc' on the vertical plane. The axis cc' does not move, cb falls on the ground line and the hypothenuse $c'b_1$ becomes parallel to $\alpha\delta_1$. Revolving the triangle back to its original position, b_1 comes in b, which is the trace, on the ground plane, of the given line. Having now the two traces, the vertical projection can be drawn by the usual construction.

75. TO FIND THE TRACES OF THE PLANE CONTAINING THREE GIVEN POINTS OR TWO GIVEN LINES. Whether two lines or three points be given, the problem consisting in passing a plane through them is the same and consists in finding the traces of the given lines or of those joining the given points. The traces of same denomination are joined by straight lines which are the traces of the required plane.

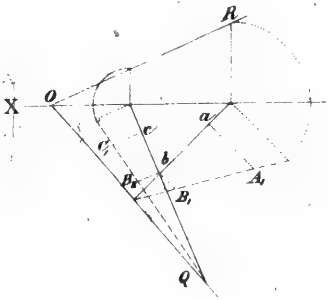


Fig.97

The traces of the lines are obtained by any of the processes of § 60,73 or 74.

In Fig.97, the heights of the three points A, B and C are supposed to be known and the traces are determined by revolving the projecting planes on the ground plane around the horizontal projections *ab* and *bc*; (§ 73). QOR is the required plane.

Sometimes the traces of the plane are required on the picture and principal plane. Revolve the principal plane around its trace Rp, Fig.98, on the picture plane, the front part of the principal plane turning to the left. The station will come in S.

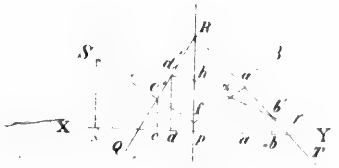


Fig.98

Let α, β, γ, be the perspectives of three points, of which the projections on the ground plane are given, a and c the traces on the picture and principal planes of the horizontal projection corresponding to α, β; d and b those corresponding to $\alpha \gamma$.

Produce $\alpha\beta$ to the intersection f with the principal line; f is the perspective of the trace on the principal plane, of the line of space corresponding to the perspective $\alpha\beta$, therefore the trace on the revolved principal plane is on Sf. But the trace is on the vertical of c, therefore it is at c'. The trace of the other line is found in a similar manner at d' and the trace of the plane containing the two lines is $c'd'$.

The traces of the two lines on the picture plane are obtained in a' and b' as in § 73 and joined to give the trace of their plane on the picture plane. The result is the plane QRT.

76. GIVEN THE LINE OF GREATEST SLOPE, TO FIND THE TRACES OF THE PLANE. The line of greatest slope of a plane is perpendicular to the trace on the ground plane. Hence to draw the traces of the plane, find those of the line and through the ground plane trace a, Fig. 99, draw aQ perpendicular to the horizontal projection, ab, of the line: it is the ground plane trace of the required plane. The trace of the plane on the picture plane is obtained by joining Q to the vertical trace, β, of the line.

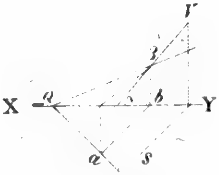


Fig. 99

In Fig.99, the line of greatest slope is supposed given by its horizontal projection, ab, and its perspective $\alpha\beta$: The traces are found by the method in § 73. Should the line be known by the heights and perspectives of two of its points or by the heights and horizontal projections, or by its slope, the traces could be determined by the methods given in § 60, 73 and 74.

77. CHANGE OF GROUND PLANE. A change in the ground plane does not produce any change in the points or lines of the ground plan: the traces of planes are displaced but remain parallel to the original trace.

Fig.100 shows the ground line moved from XY to $X_{,}Y_{,}$; the left hand figure contains the projections of a point, of a line, and the traces of a plane before the change of ground plane.

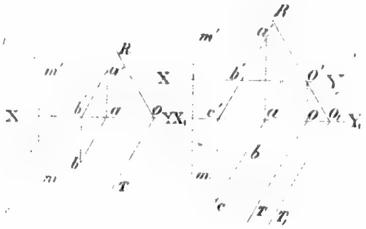


Fig.100

In the first place, it may be observed that there

is no change in the vertical plane, beyond moving the ground line from XY to X_1Y_1.

In the ground plane, the projections of the point, *m*, and of the line, *ab*, remain the same, but the trace of the line is now in *c* instead of *b*. The new trace is obtained by producing the vertical projection, *a′b′*, across the old ground line, XY, to the new one, erecting the perpendicular *c′c* and producing *ab* to its intersection with *cc′*.

The trace of the plane has been moved from OT to O_1T_1. To find the new one, produce the vertical trace $O'R$ across the old ground line XY to the new one, X_1Y_1, and through the point of intersection, O_1, draw O_1T_1 parallel to OT.

78. TO FIND THE HORIZONTAL PROJECTION OF A FIGURE FROM ITS PERSPECTIVE WHEN THE FIGURE IS CONTAINED IN A PLANE PERPENDICULAR TO THE PRINCIPAL PLANE. Take for vertical plane of projection the principal plane and let QZ Fig. 101. be the trace of the plane containing the figure. Take for ground plane the horizontal plane passing through the point of intersection of QZ with the trace QR of the picture plane, XY will

be the ground line. Let S be the station, s the foot of the station, n, n', a point of the given figure and m, m', its perspective. The given plane, being perpendicular to the principal plane, the vertical projection of any point of the former is on the trace QZ. The picture plane, RQT, is perpendicular to both planes of projection, therefore the projections of any point of the picture plane are on its traces.

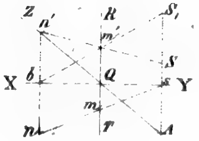


Fig. 101

Produce QZ to the intersection A with the vertical of the station and take $SS_{,}$ equal to sA, $S_{,}$ being above or below S according as A is below or above s. Join $S_{,}m'$ and produce to the intersection b with the ground line: join Sn' and $n'b$. The line Sn' passes through m', since m' is the perspective of n'.

The similar triangles $n'm'Q$, $n'SA$ give:-

$$\frac{Qm'}{SA} = \frac{n'Q}{n'A}$$

From the triangles bQm', $bsS_{,}$, we have:

$$\frac{Qm'}{sS_{,}} = \frac{bQ}{bs}$$

But $$SA = sS_1$$

Therefore: $$\frac{n'Q}{n'A} = \frac{bQ}{bs}$$

or, $$\frac{n'Q}{QA} = \frac{bQ}{sQ}$$

Hence the triangles $n'bQ$, QsA are similar, as having one angle equal and the sides about it proportional, consequently bn' is parallel to sA or perpendicular to XY and the point n is the trace on the ground plane of the visual ray sn, S_1b. Were the eye placed in S_1 the point of the ground plane which would be found to correspond to mn' of the perspective is the horizontal projection of the point of the plane QZ. Should the new station S_1 be used in connection with the perspective of a figure in the plane QZ, the result obtained, when constructing the corresponding figure of the ground plane would be the horizontal projection of the figure in the plane QZ.

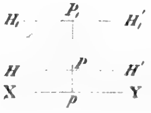


Fig. 102

Therefore to obtain the horizontal projection of a figure in a plane perpendicular to the principal plane, take for ground line the trace XY,Fig.102,of the given plane on the picture plane,find the height of the station above the point of intersection of its vertical by the given plane (§ 75) use it as height of the new station and draw the horizon line $H_{,}H'_{,}$ on the perspective at that height above the ground line. The figure constructed from the perspective by any of the methods of § 65, 66,67 will be the horizontal projection of the figure in space.

It has been shown that the perspective is the same as if the horizontal projection had been seen from the station $S_{,}$ instead of observing the original figure (Fig.101),from S,consequently,the precision of the result (§ 70) is increased in the proportion of $\frac{sS_{,}}{sS}$ by the inclination of the plane of the figure. Were the plane falling instead of rising in front of the observer, $sS_{,}$ would be smaller than sS and the precision would be decreased.

Hence a perspective taken for the purpose of constructing a figure in an inclined plane should always be taken in the direction of the rising plane; thus a river at the bottom of a sloping valley should be taken looking up the valley.

79. TO FIND FROM ITS PERSPECTIVE THE HORIZONTAL PROJECTION OF A FIGURE IN A PLANE PERPENDICULAR TO THE PICTURE PLANE. The method of squares of § 67 can be applied to a figure in any inclined plane, by conceiving vertical planes containing the sides of the squares. The intersections of these planes by the inclined plane form a series of parallelograms corresponding to the squares of the ground plane.

Let QR, Fig. 103, be the trace on the picture plane of a plane perpendicular to it, XY the ground line, P the principal point, and *abcd* one of the squares of the ground plan. The projecting planes of *ab* and *cd* cut the trace QR in *m* and *n*. Through the station, S, draw a parallel to the intersection of the projecting planes with the plane QR: the horizontal projection *st* is parallel to *ab* and

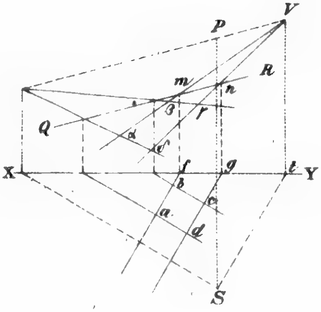


Fig. 103

cd; the vertical projection passes through P and is parallel to QR, since all lines in the plane QR are projected vertically on QR. At *t* erect the perpendicular *t*V to the ground line: V is the vanishing point of the intersections of the projecting planes with the plane QR and the lines V*m* and V*n* are the perspectives of these intersections. The distance *mn* can be carried on QR and as many parallels placed in perspective as necessary.

The same operation is repeated for *ad* and *bc*, and the figure $\alpha\beta\gamma\delta$ obtained on the perspective corresponds to the square *abcd*.

Another process consists in constructing the figure in the inclined plane by one of the methods of § 65, 66, 67, using the plane of the figure as ground plane (§ 58).

Let QR Fig. 104 be the trace of the plane of the figure on the picture plane, HH′ the horizon line and P the principal point.

To construct the figure in the plane QR, that line is taken as ground line: the new horizon line is a parallel H_1H_1' to QR through the principal point.

The height of the station is the distance of these two lines, $Pp_{,}$. The line which will appear as the projection of the principal line on the constructed figure will be the perpendicular to the picture plane at $p_{,}$. On the real ground plane, the distance between the two projections of the principal line will be equal to $pp_{,}$.

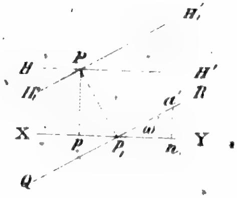


Fig. 104

Having obtained the figure in the plane QR, let us now take for ground plane the horizontal plane of $p_{,}$, the ground line being XY.

Let ABCD, Fig. 105, be the figure in the plane QR, $p_{,}t$ the projection of the principal line, QR the trace of the picture plane and $s_{,}$ the foot of the station. The projection of A on the ground plane is at the same distance from the ground

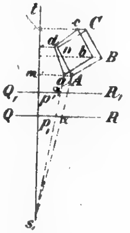


Fig. 105

line as A is from QR, but the distance of the projection, from $s_,t$ is equal to mA multiplied by the cosine of the inclination ω of the plane QR, for let a', Fig. 104, be the vertical projection of A and the right angle triangle $p_,a'n$ gives:

$$p_,n = p_,a' \cos.\omega .$$

Therefore if Am, Fig. 105, be drawn parallel to QR, am taken equal to Am cos.ω and the same operation repeated for B, C, and D, the resulting figure $abcd$ is the ground plan of ABCD.

The ground plan may be obtained in another way, for, join $s_,$A: the intersection α with QR is the projection on QR of the point of the perspective corresponding to A. Take $s_,p'$ equal to $s_,p_,$ sec.ω and through p' draw $Q_,R_,$ parallel to QR; join $s_,a$. The similar triangles $s_,p_,\alpha$, $s_,m$A, give:

$$\frac{s_,p_,}{p_,\alpha} = \frac{s_,m}{mA} \qquad (1)$$

From the similar triangles $s_,p'\alpha_,$, $s_,m$A, we have:

$$\frac{s_,p'}{p'\alpha_,} = \frac{s_,m}{ma} \qquad (2)$$

Dividing (1 (by (2), replacing $s_,p'$ and ma by $s_,p_,$ sec.ω and mA cos.ω respectively, we find :

$$p'x_1 = p_1x.$$

This means that if the perspective be moved in QR_1, the directions obtained from the perspective for the different points of the plane QR will be the directions of the horizontal projections of these points.

Therefore to construct the horizontal projection of the figure seen in perspective, find the distances of the various points of the figure from the picture plane by means of a vanishing scale (§ 68) made with Pp_1, Fig. 104, as height of the station and the real distance line. Then find the directions of the projections, using QR as ground line and a distance line increased in the proportion $\frac{1}{\cos.\omega}$. The figure constructed with the above distances and directions will be the horizontal projection of the figure in the plane QR.

80. CHANGE OF GROUND PLANE AND DISTANCE LINE. Let A, Fig. 106, be a point of a figure in a plane perpendicular to the picture plane and β its perspective. Take the plane of the figure as ground plane and let Sp_1, be the trace of the assumed principal plane. Revolve the principal plane around its trace on the

ground plane: the station will come in S, b and b' are the projections of a and a' the projection of A on the assumed principal plane. Move the perspective to $b_1 b_1'$ so that $s_1 p' = \frac{s p}{\cos . \omega}$, ω being the angle of the assumed and real ground planes; it has been shown that $s_1 b_1$ is the direction of the projection a of A on the real ground plane supposed to be revolved around $s_1 p_1$ on the assumed ground plane. The visual ray, however, does not pierce the ground plane in a_1, its projection on the principal plane having been changed from Sb' to Sb_1' by the displacement of the perspective. But join Sp' and take as new ground plane the plane passing through c : $s'a_1'$ will be the trace of the assumed principal plane on the new ground plane and a_1' the projection of the trace of the visual ray on the last plane. Consequently the trace of the visual ray is at the intersection of $s_1 a$ with the perpendicular drawn from a_1' to $s_1 a'$.

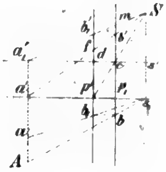


Fig. 106

Similar triangles give the following proportions:

$$\frac{p'a'}{b'b_1'} = \frac{p'f}{f b_1'} = \frac{c b'}{b'm} = \frac{d b_1'}{b'm} = \frac{d a_1'}{b'b_1'}$$

or

$$\frac{p'a'}{b'b_1'} = \frac{d a_1'}{b'b_1'}$$

and

$$p'a' = d a_1'$$

$p'a'$ being equal to $d a_1'$ the figure $p'd a_1' a'$ is a parallelogram and $a_1'a'$ is perpendicular to $s_1 a'$, therefore the visual ray will pierce the new ground plane in a'.

Hence, if the perspective be moved from p_1 to p' and $s a_1'$ taken as ground plane, the perspective viewed from the station will correspond on the assumed ground plane to the projection of the figure on the real ground plane: this projection can consequently be constructed by the methods of § 65, 66, 67.

Fig. 106 gives the proportion:

$$\frac{Ss'}{Ss_1} = \frac{s'c}{s_1 p'} = Cos.\omega$$

or

$$Ss' = Ss_1 \, Cos.\omega$$

The heights Ss', Ss_1, of the station above the various ground planes being the same as the distance of the principal point from the corresponding ground lines, the new ground and distance lines can be found as follows:

Let QR, Fig. 107 be the trace on the picture plane of the plane containing the figure. From the principal point P, let fall Pp, perpendicular to QR and draw Pp and pp perpendicular and parallel to the real horizon line HH'. Take Pd equal to Pp and draw Q, R, parallel to QR: it is the ground line to be used in the construction, because

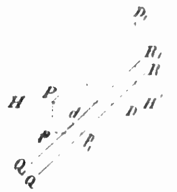


Fig. 107

$$Pd = Pp, \cos \omega.$$

At the distance point, erect DD, perpendicular to HH' and draw PD, parallel to QR; PD,, is the length to be used as distance line.

The height of the station Pd used for the construction is always smaller than the real height Pp, above the plane of the figure, therefore the precision of the construction is less than if the figure had been in a horizontal plane.

81. FROM THE PERSPECTIVE OF A FIGURE IN ANY GIVEN PLANE, TO CONSTRUCT THE HORIZONTAL PROJECTION OF THE

FIGURE. The method of squares can be again employed in this case. Let QOR, Fig. 108, be the traces of the plane of the figure on the ground and picture planes, and *abcd* one of the squares of the ground plan. The projecting plane of *ad* intersects the traces of QOR in Q and L: the vertical projection of the intersection of the two planes being L*q*′. Through the station draw a parallel to *ad*, L*q*′: the horizontal projection is *sv* parallel to *ad* the vertical projection is PV parallel to L*q*′ and the vertical trace, V, is the vanishing point of the intersection of QOR with the projecting plane of *ad*. The perspective of this intersection is VL: the perspectives of the intersection of the projecting plane of *cb* is VK and all the lines required may be drawn in perspective by carrying the distance LK on the trace OR and joining the points of division to V.

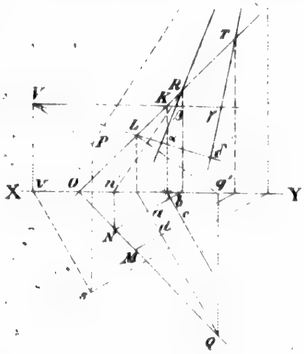


Fig. 108

The perspectives of the intersections with the plane QOR, of the projecting planes of ab and cd are obtained in a similar manner by drawing through the station a parallel to ab, n'R, for instance, and joining the vanishing point V′ to R and T. The resulting figure $\alpha\beta\gamma\delta$ corresponds, on the perspective, to the square $abcd$ of the ground plan.

It is also possible to construct a vanishing scale (§ 68) so as to find the distances of the various points from the picture plane.

Through the station, a plane is drawn perpendicular to the vertical trace of the given plane: the intersections of the latter with the picture plane and the station point are placed in their actual positions and the vanishing scale is constructed by measuring the equal distances from the trace of the picture plane.

82. **CHANGE OF STATION, GROUND AND PICTURE PLANE.** The same result is arrived at by changes in the relative positions of the station, perspective and ground plane.

Let QR, Fig. 109, be the trace on the principal plane, of the plane containing the figure, which we

will call A. Take for ground plane the horizontal plane passing through the intersection *p* of this trace with the principal line and suppose the principal plane revolved around its trace *sp* on the ground plane.

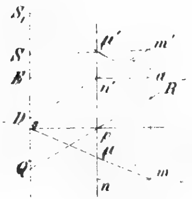


Fig. 109

Let S be the station and $\mu\mu'$ the perspective of the point *mm'*, in the plane A. Take SS, equal to Q*s* and suppose that S, be used as station in connection with a new plane passing through *sp* and the trace on the picture plane of the plane A. Call this plane B. The visual ray from the new station to μ,μ', will be projected in $S_{,}\mu'$, $s\mu$.

Cut the planes A and B by a third parallel to the principal plane and passing through the point *mm'*.

The horizontal projection of both intersections is *mn*, parallel to *sp*. The projection on the principal plane of the intersection with plane A is *m'n'* parallel to QR and the intersection with plane B is projected in $n'a$ parallel to sp.

Join $S\mu'$ produce to m', produce $S_1\mu'$ to its intersection with $n'a$; $m'n'$ and $n'a$ to their intersection with S_1Q. Join $m'a$. The similar triangles give:

$$\frac{n'\mu'}{SD} = \frac{m'\mu'}{m'S} \qquad (1)$$

$$\frac{n'\mu'}{S_1E} = \frac{a\mu'}{aS_1} \qquad (2)$$

But

$$SD = SE + ED = SE + sQ = SE + S_1S = S_1E$$

hence the first terms of (1) and (2) are identical and we have:

$$\frac{m'\mu'}{m'S} = \frac{a\mu'}{aS_1}$$

which is transformed into:

$$\frac{m'\mu'}{\mu'S} = \frac{a\mu'}{\mu'S_1}$$

The triangles $SS_1\mu'$ and $am'\mu'$ having one angle equal and the two sides about it proportional, are similar and $m'a$ is parallel to SS_1. Consequently a is on the perpendicular $m'm$ to sp.

The line sm, S_1a is the visual ray from the new station through the point $\mu\mu'$ of the perspective: mn, an', is a line of the plane B. These two lines intersect since the intersections m and a of their projections are on the same perpendicular to the ground line and the point of intersection is the trace of the visual ray on the plane B since the line mn, an' is in that plane. The same point is also the trace on the plane B of the vertical from mm'.

Therefore if verticals be let fall from all the points of the figure in plane A, their traces on plane B will form a new figure which will correspond to the perspective viewed from S_1.

The problem is thus reduced to construct from its perspective the horizontal projection of a figure contained in a plane perpendicular to the picture plane, which is done by a change of ground and picture planes (§ 78). The process now involves changes of station, ground plane, picture plane and trace of principal plane as follows:-

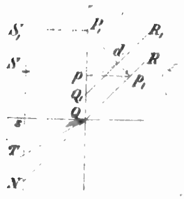


Fig.110

Let QP_1, Fig.110, be the principal line. Revolve the principal plane on the picture plane around QP_1, the front part of the principal plane being turned to the left: the station comes in S, and NS_1 is the vertical of the station. Let TQR be the plane containing the figure seen in perspective. Draw Qs perpendicular to QP_1 and take SS_1 equal to sT. Draw S_1P_1 parallel to sQ. The point P_1 is to be used as principal point of the perspectiv[illegible]

Draw P_1p_1 perpendicular to QR, pp_1 parallel to sQ and take P_1d equal to P_1p. Through d draw Q_1R_1 parallel to QR; it is the assumed ground line.

Produce QR to N: QN is the length to be assumed as distance line.

On the constructed figure, the perpendicular to the picture plane at p_1 will appear as trace of the principal plane on the ground plane.

The traces of the plane containing the figure are found as in § 73.

83. REFLECTED IMAGES. The case of horizontal reflecting surfaces is the only one that will be considered.

When a perspective contains the direct and reflected images of the same point, the point can be located in space, provided the altitude of the station above the reflecting surface be known.

Take for ground plane the reflecting surface and revolve the principal plane on it, around its trace. Let a, a', Fig. 111, be the point in space, α, α' its perspective and $\alpha\alpha'_1$ the perspective of its reflected image. The horizontal projection is the same for both images, because the reflecting surface being horizontal, the direct and reflected visual rays are in the same vertical plane having for trace sa.

Let sa, SOa', be the reflected visual ray: according to the laws of reflection, the direction of SO is the same as if a' were placed at a distance equal to ca' below the reflecting surface and on the same vertical.

Produce $a'O$ to S_1: cb being equal to ca', sS is equal to sS_1. Hence, to find the position in space of a, a', take sS_1 equal to sS: join $S\alpha'$, $S\alpha'_1$ and S_1O; the point of intersection of $S\alpha'$ and S_1O is the vertical projection of the point of space.

Join sa and produce to the intersection with *a'a*, perpendicular to the ground line; *aa'* is the required point.

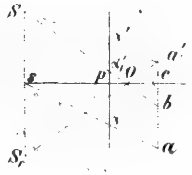


Fig.111

The construction gives not only the position *a* of the point on the ground plane, but also its height *ca'*.

The middle of the vertical between the direct and reflected images corresponds to *a*, the horizontal projection of the point on the ground plane. This shows that when the shore of a lake, for instance, is indistinct on a perspective, it would be incorrect to take for shore line, the middle line between objects and their images in the lake, because this would give for distance of the shore that of the objects themselves.

84. SHADOWS. The subject of shadows is an important branch of perspective, but only those cast by the sun need be considered here.

Let α and β, Fig.112, be the perspectives of two points A and B, *m* and *n* their shadows. The line joining A to its shadow is the direction of the sun and so is the line joining B to its own shadow; therefore these lines are parallels and their vanishing point is V, at the intersection of $m\alpha$ and $n\beta$.

A line drawn from the station to the sun is parallel to the first two, because it is also the direction of the sun; therefore V is its trace on the picture plane or the perspective of the sun.

From V let fall Vv perpendicular to XY; sv is, on the ground plane, the direction of the sun. On the horizon line, take CF equal to sv and join FV: FVC represents, revolved on the picture plane around VC, the triangle having its vertex at the station and VC as opposite side. Therefore VFC is the altitude of the sun. Having the sun's altitude, the azimuth of the line sv of the ground plan can be calculated, provided the latitude and approximate time are known.

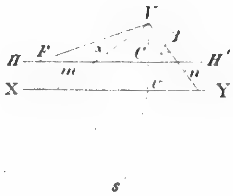


Fig. 112

Fig. 112 represents the sun in front of the observer. When it is behind, the line between the station and the sun does not pierce the picture plane: it has to be produced to intersect it below the horizon line. The trace of this line on the picture plane V, Fig. 113, is still considered as the perspective of the sun: it is obtained in the same manner as when the sun is in front and all demonstrations apply to one case as well as to the other.

The calculation of the azimuth can be made by the method given in § 37 for the solution of spherical triangles.

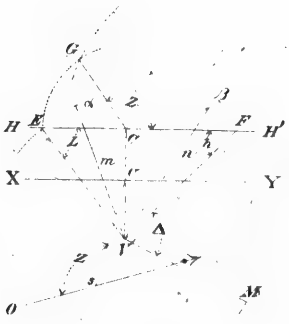


Fig. 113

Find the altitude CFV of the sun by the construction given above, make EVC equal to the colatitude of the place and FVM, to the polar distance of the sun. Take VM equal to VE and from C and F as centers with CE and FM respectively as radii, describe arcs of circle. Join their point of intersection, G, to C, and GCF is the azimuth of the sun.

When the perspective has been taken in the morning, plot the angle Z on the left of sv in vsO, and the line Os is the north and south line of the ground plan.

In the afternoon, the angle Z should be plotted on the right of sv. The rules are reversed when the perspective of the sun is above the horizon line.

85. HEIGHTS. In general, one perspective is not sufficient to determine the height of a point, although there are exceptions as for instance, points on the horizon line which are at the same height as the station.

The horizontal projection of the point being known, the height above the ground plane is measured with a scale in the same manner as a vertical is divided into equal parts (§ 60).

For instance α and a, Fig. 114, being the perspective and horizontal projection of a point, and s the foot of the station, draw aF parallel to XY.

From the trace p of the principal line, take pB equal to the distance of a from XY. Join sB, and FE is the height of the point above a.

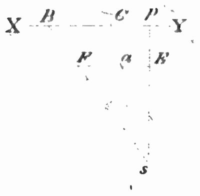


Fig. 114

This height being a fourth proportional to three known lines, can be found with an ordinary sector. Take with a pair of compasses the distance from a to XY, place one of the points on the division p of the sector (Fig. 115) which expresses the length of the distance line, and open the sector until the second point of the compasses coincides with the corresponding division of the other branch, sp and sB being equal. Now take with the compasses the distance from a to XY (Fig. 114) and place one of the points in p (Fig. 115). The other point being placed on sp, will

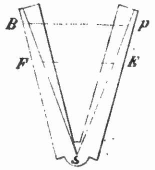


Fig. 115

coincide with a division of the scale,E for instance; then turn round the compasses and take the distance from E to the same division of the scale sB:it is the height of the point above the ground plane.

CHAPTER III

7

PERSPECTIVE INSTRUMENTS

86. **Many** instruments have been devised for producing at once a perspective, either by mechanical or optical means.

One of the simplest forms is probably the wire grating represented in Fig.116. Wires are stretched on a frame so as to divide it into small squares. The frame is placed in front of the object or view to be reproduced

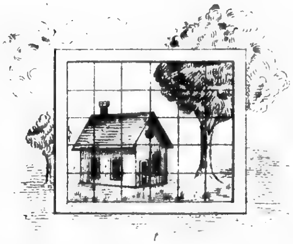


Fig.116

and the draughtsman looks through an eye-hole in a fixed position. Dividing his paper into squares in the same manner as the frame, he is able to reproduce the outlines of the subject by drawing his lines through the squares of the paper corresponding to those of the frame. The distance from the frame to the eye-hole is the distance line of the perspective when the squares of the paper are equal to those of the frame.

87. DIAGRAPH. While for artistic purposes, the grating is quite sufficient, there is some uncertainty in drawing the figures of the corresponding squares. To obviate this defect, it has been imagined to follow the outlines of the subject with a pointer moved by the hand, as in Fig. 117.

A drawing board, on which is stretched a piece of paper, is placed in an upright position in front of the subject of the perspective. It is provided with a straight edge, supported at both ends by cords attached to a counterpoise at the back of the board. The straight edge may be moved up and down or right and left, but owing to the mode of suspension, is always parallel to the same direction. The middle of

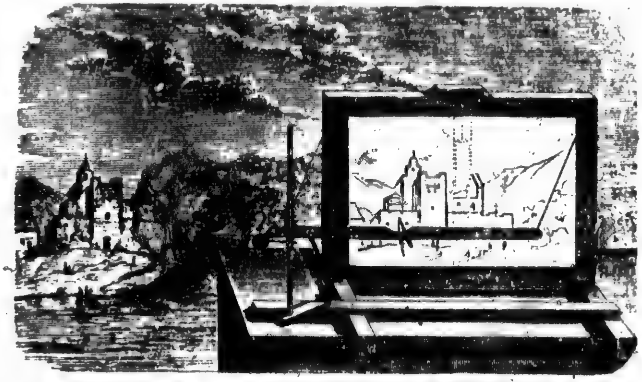


Fig. 117

the straight edge carries a pencil resting on the paper and the end has a pointer corresponding to the eye-hole of the instrument. The draughtsman takes the pencil with the hand and placing his eye at the eye-hole, he follows with the pointer the outlines of the subject, by moving the straight edge in the proper direction. The pencil reproducing exactly the motion of the pointer, describes the perspective on the drawing board.

The plane in which the pointer moves is the picture plane, the eye-hole is the station and its shortest distance from the pointer is the distance line.

The upright position of the drawing board is inconvenient: in a modification of the same instrument called "Diagraph", the paper is placed horizontally and the motion of the pointer is transmitted to the pencil by a cord and pulley.

None of these instruments, however, have come into general use.

88. CAMERA LUCIDA. The camera lucida consists essentially of a four sided prism having a right angle, two angles of 67° 30' and one angle of 135° (Fig. 118). The eye is placed directly above the edge of the prism so that the pupil receives at the same time the rays of light emitted by objects placed in front and those coming from the surface of the paper. It is therefore

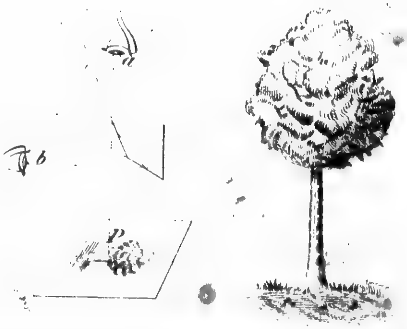


Fig. 118

possible to follow with a pencil on the paper the outlines of the subject, the point of the pencil being seen directly and the subject by a double reflection. The position of the eye must not vary during the operation; to secure this, the upper face of the prism is covered by a metal plate with a small hole cut in the edge, through which the draughtsman has to look.

With the form of instrument just described, the eye receives simultaneously impressions from objects placed at different distances; the pencil is quite close while the subject of the perspective is generally far away. The eye cannot accommodate itself to both distances at the same time; one of the images is always more or less confused and the work is very trying to the eyes.

In the more refined instruments, the upper surface of the prism is ground in the form of a concave lens, giving to the reflected rays the same convergence as if emitted by an object twelve inches away. The paper being placed at the same distance below the prism, the pencil and the subject appear now at the same distance and can be seen simultaneously without any effort.

The center of the pupil, *a*, Fig.118, is the station point of the perspective drawn in P, and the height of *a* above the plane of P is the distance line. But, the subject if looked at directly from *a* would not appear as represented; it would be necessary to move the eye to *b*, virtual image of *a* with reference to the two reflecting surfaces of the prism.

A well constructed camera lucida is provided with colored glasses, to equalize the brightness of the images.

89. CAMERA OBSCURA. A camera obscura, in its simplest form, is a box hermetically closed to extraneous light, except that coming through a lens placed on one of the sides. The opposite side of the box being in the focal plane of the lens, an image is formed on it of the distant objects situated in front of the lens.

Making abstraction of the errors introduced by lenses, the image of the camera obscura is a true perspective, for it is the same as would be drawn on a picture plane placed in front of the lens at a distance equal to the focal length.

Let O, Fig. 119, be the optical center of the lens and OA, OB, OC, three rays of light coming from three distant points of space. The images of the points will form a triangle ABC on the focal plane R of the camera. The same rays of light will form, by their intersection with a plane Q parallel to R and at the same distance from O, another triangle *abc* in which every side is equal and parallel to the corresponding side of ABC, therefore, *abc* = ABC.

Fig. 119

The same demonstration applies to any other triangle and as a figure can always be decomposed into a number of triangles, any figure obtained on the plane R will be the same as on the plane Q reversed. But the figure on Q is the perspective seen from the station O on the picture plane Q, therefore the image of the camera obscura is the perspective seen from the optical center of the lens on a picture plane placed at the first focus. The focal length of the lens is the distance line.

Before the invention of photography the camera obscura was used for drawing perspectives and various forms were devised to adapt it to that purpose.

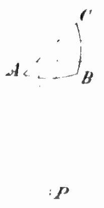


Fig.120

One form consists of a prism ABC, Fig. 120, with two spherical faces AB and BC, and a plane reflecting face AC. The parallel rays of light emitted by the subject are brought to a focus by the two spherical faces AB and BC, while they are reflected at right angles by the face AC.

The prism is placed on the top of a tripod which supports a drawing board at the proper height to receive the image formed at the focus of the object glass. The tripod is then covered with a black cloth to shut off extraneous light and enable the draughtsman to see the image projected on the paper and follow it with a pencil.

The point of the pencil being between the lens and the paper, casts a shadow just at the point where the image is wanted. The instrument shown in Fig.121 is not open to the same objection and requires, neither tripod, drawing board nor even a black cloth. It is merely a box with a lens in front and a mirror in PQ inclined at 45° to the axis of the box. The image formed by the lens is reflected by the mirror on a ground glass placed in AB. Being inverted a second time by the reflection, it appears now upright. The lid which covers the ground glass when not in use, is open at an angle of about 45° with the ground glass and cuts off sufficient light for the image to be visible. In these conditions, the image is not bright enough to work on paper and has to be traced on the ground glass, but with a black cloth covering the box and the head of the draughtsman, it is possible to work through thin paper.

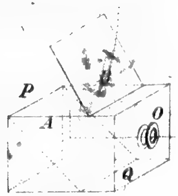


Fig.121

90. PERSPECTOGRAPH. The perspectograph is the

invention of Hermann Ritter a German Architect (1) its object being to draw a perspective from the plans of the subject and not from the subject itself. The lines of the plans are followed with a tracer and the perspective is drawn by a pencil carried by another part of the instrument.

As constructed by Chs. Schröder & Cie (Frankfort-on-the-Main), it is a large instrument made partly of wood, partly of metal, well adapted for drawing perspectives of buildings from an architect's plans, but useless for drawing a topographical plan from a perspective.

For surveying purposes, the instrument should be of small size, made entirely of metal; all the parts should be fitted with precision It should work easily and the amount of dead motion should be as small as possible.

Although in its present form it cannot be employed in connection with photographic surveys, its

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(1) Perspectograph, von Hermann Ritter, Architekt, Frankfurt a.M. Druck von J. Maubach & Co.

K

PERSPECTOGRAPH

theory is given at length: with some slight modifications, it is applicable to any other perspective instrument.

However perfect an instrument may be, it always introduces in the final result some errors of its own due to dead motion, to imperfections in the adjustments, and to the slight errors unavoidable in the determination of the constants. Whenever the precision of the survey requires it, a geometrical construction should be employed, and the use of perspective instruments should be restricted to reconnaissance or rough surveys, in which rapidity is more important than perfect accuracy.

The principle of the apparatus is as follows: Let S, Fig. 122, be the station, M a point of [illegible] ground plane and μ its perspective. Take sS_1 parallel to the ground line and equal to sS: join S_1M.

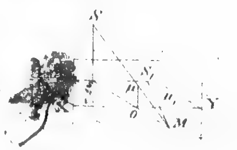


Fig. 122

The similar triangles sSM and $O\mu M$ give

$$\frac{sS}{O\mu} = \frac{Ms}{MO}$$

From the triangles sS_1M and $O\mu_1M$, also similar, we have:

$$\frac{sS_1}{O\mu_1} = \frac{Ms}{MO}$$

Hence,

$$\frac{sS}{O\mu} = \frac{sS_1}{O\mu_1}$$

But, by construction, sS is equal to sS_1, therefore

$$O\mu = O\mu_1$$

This property furnishes a new method for constructing a perspective. Take on the ground plan, Fig. 123, sS_1 parallel to the ground line and equal to the height of the station. To find the perspective of a point M of the plan, join sM and S_1M, which intersect the ground line in O and μ_1. On another part of the paper, draw the ground line of the perspective, $X'Y'$ and take on it a point O' to represent the point O of the ground plan. At O' erect $O'\mu$ perpendicular to $X'Y'$ and equal to $O\mu_1$; μ is the perspective of M. Owing to the position of the figure, the

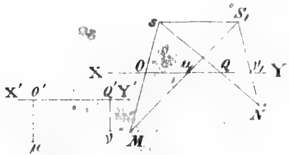


Fig. 123

perspective appears upside down.

The perspective of another point, N, of the ground plan is obtained in a similar manner, by taking OQ' equal to OQ and $Q'\nu$ equal to $Q\nu_1$.

This is done mechanically by the perspectograph: sM and S_1M, Fig. 124, are two wooden arms jointed in M and carrying the tracer. They slide through four adjustable pieces; s, S_1, O and μ_1: s and S_1 can be adapted to any part of a rule RT. s is fixed at the point of the ground plan representing the foot of the station and the rule or slide RT is firmly fixed to the drawing board, parallel to the ground line. The second piece, S_1, is placed at a distance of s equal to the height of the station and fixed in that position.

The third piece, O, is fixed to a rod which moves in the groove of a slide XY and carries a pantograph system with axis at D fixed to the rod, so that the distance from O to D is invariable while the

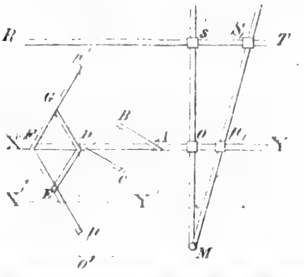


Fig. 124

instrument is in use. When the arm sM is moved, s being a fixed point, O follows the motion of the arm and carries with it along the groove XY the moveable rod and the pantograph system.

The fourth piece μ is connected with the joint A of the pantograph system, so that the distance μA is invariable during the operation; it is also bound to slide on the moveable rod.

The pantograph system, is composed of four straight arms, AB, AC, Fμ and Fμ', and two arms CDE and BDG, bent at right angles in D. They are joined in A, B, C, D, E, F and G, the sides of the parallelograms ABDC and DGFE being all equal. The arms Fμ and Fμ' are double the length of one side of the parallelograms and the pencil which is to describe the perspective may be placed either in μ or μ'.

The sum of the four angles at D is equal to four right angles: two of these angles, CDE and BDG, being right angles, the sum of the two remaining ones must be equal to two right angles, that is,

$$CDB + EDG = 180^\circ.$$

But in a parallelogram the sum of two adjacent angles is equal to two right angles, that is,

$$CDB + DCA = 180^\circ$$

hence,

$$EDG = DCA.$$

Therefore the two parallelograms are equiangular and their sides being equal, the parallelograms are equal, but not placed in the same direction. The diagonal DA of one is equal to the diagonal GE of the other, and BC is equal to DF.

The line $\mu\mu'$ is parallel to GE because $F\mu$ is equal to $F\mu'$; it is therefore perpendicular to XY since the diagonals of a rhombus intersect at right angles, and it passes through D, because $E\mu$ is equal and parallel to GD. We have also

$$D\mu = GE = DA.$$

It is now easy to understand the working of the instrument.

The slide XY is placed on the ground line or rather on the line representing the trace of the picture plane on the ground plan. When the tracer in M is moved on a parallel to XY, the arm $M\sigma$ carries with it the moveable rod and the pantograph system attached. The distance from O to μ_1 does not vary, since the similar triangles MsS_1 and $MO\mu_1$ always give the same proportion between $O\mu_1$ and the constant length sS_1. The distance from μ_1 to A and from O to D being invariable, and $O\mu_1$ being constant, AD and consequently $D\mu$ do not change, and the pencil in μ describes a parallel to XY: it is the perspective of a line of the ground plane, parallel to the picture plane.

When the tracer is moved away from XY, in the direction of Ms, the points O and D do not change, but $O\mu_1$ is lengthened and μ_1 moves towards the right carrying with it the joint A and increasing the diagonal DA to the same extent as $O\mu_1$; $D\mu$ being equal to DA, will also be lengthened and μ moves down, precisely the same distance as μ_1 moved towards O.

The construction thus effected mechanically is the same as in Fig. 123. The ground line of the perspective, XY, is the line which would be described by the pencil in μ, if the tracer M could be brought to the center of the groove and moved on XY: O and μ_1 would then coincide.

Drawing the tracer away from XY, but in the direction sM, μ, separates from O, and μ moves down by the same quantity, from its former position O' on the ground line, $O'\mu$ being perpendicular to $X'Y'$ and

$$O'\mu = O\mu,$$

Now if M be placed on any other point of the ground plan, the perpendicular $D\mu$ to $X'Y'$ will be carried away the same distance as the point O, and μ will be at a distance from $X'Y'$ equal to the new value of $O\mu$, .

The perspective is upside down, the draughtsman having to place himself near M for guiding the tracer.

The end μ' of the arm $F\mu'$ describes the symmetrical figure of the perspective, or the image which would be seen in a mirror, but were the fixed point S, placed on the left of s, μ, would describe the true perspective, the direction of the motions of μ and μ' being reversed. The ends μ and μ' of the arms of the pantograph system are both fitted to receive the pencil which can be changed from one end to the other as required.

The instrument set as in Fig. 124 can only work on points or figures beyond the picture plane: it is possible to place the slide XY on the other side of RT, so as to work on points between the picture plane and the station, but the obliquity of the arms prevents them from sliding freely and the working of the instrument is unsatisfactory.

It may happen that with a high station and points at the extreme right or left of the station (it would be the extreme left on the figure) the obliquity of the arm S_1M becomes too great to work. S_1 must then be changed from one side of s to the other, (from the right to the left of s on the figure) and the perspective from one side of the ground line to the other. The pencil is at the same time changed to the opposite arm of the pantograph system.

The sliding rod XY may be reversed end for end in its groove, the pantograph system coming on the opposite side of the movable arm Ms. The pencil does not require to be changed, but the arm bearing it, μ for instance, instead of being between RT and XY, will now be on the other side of XY.

A scale must be drawn on RT, the zero corresponding to the pointer carried by the sliding piece *s* : the graduation extends to right and left and the pointer of the sliding piece S, is set opposite the division corresponding to the height of the station above the ground plane.

The distance between RT and XY is equal to the distance line of the perspective.

In the case of figures in planes which are not perpendicular to the principal plane, it has been shown that the solution of problems involves changes in the distance line: the edges of the drawing board must therefore carry graduations permitting to move XY by any given quantity, keeping it parallel to RT

The different pieces of the instrument are adjustable and must first be placed in proper position for the work in hand. This done, the slide RT is firmly fixed to the drawing board and XY placed parallel to RT. There remains now to draw the scales and determine the position of the various lines and points on which rests the construction of the perspective Hitherto it has been assumed that the points *s*, S, O, *μ*, etc, were mathematical points and that their distances could be measured directly, but there is nothing on the instrument to define their exact position and no such measures could be taken with precision, consequently, these quantities or the constants of the instrument, have to be determined indirectly.

91. TO DRAW THE TRACE OF THE PRINCIPAL PLANE ON THE DRAWING BOARD. It is assumed that the grooves in the sides RT and XY are parallel to their edges and that the sliding motion in these grooves is also parallel to the same direction. This assumption is practically correct.

Place S, above *s* (Fig. 125) and draw a perpendicular MM, to RT, which, if produced, would pass as nearly as possible through *s*. Should this line pass exactly through *s*, it would be the trace of the principal plane on the ground plane, and were the tracer M moved along the line from M to M, , the point O of the slide XY would not move. To ascertain whether O has moved or not, mark the position

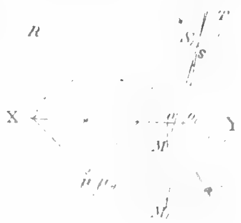


Fig. 125

of μ when the tracer is in M, then place the tracer in $M_,$ and if μ has moved to the right, in $\mu_,$, this shows that $MM_,$ is too much to the left. Should MO and $M_,O$ be equal respectively to one half and twice the distance line, the error in the position of $MM_,$ would be equal to three times the displacement of μ, but it is sufficient to estimate the quantity by which $MM_,$ has to be displaced and to repeat the trial two or three times. The motion of O is indicated by the displacement of μ to the right or to the left: a motion of μ perpendicularly to XY indicates merely that $S_,$ is not precisely over s.

92. TO FIND THE DISTANCE FROM THE STATION TO A FRONT LINE OF THE GROUND PLANE. The trace of the principal plane on the drawing board being now determined, the ground plan could be placed in its proper position on the board, were it possible to measure exactly the distance from the point s of the instrument to a front line AB, Fig. 126, of the drawing board. This determination is made as follows:- Draw a second line CD parallel to AB; place the two long arms one above the other and bring the tracer to M: mark the

position of μ. Then carry the tracer to CD and follow the line until the pencil has returned to the same point μ: the line NM, if produced, would pass through s. Repeat the same operation in $M_1 N_1$ and μ_1. Having the lengths of MM_1 and NN_1 and the distance d between AB and CD, a simple proportion gives the distance x from s to AB:

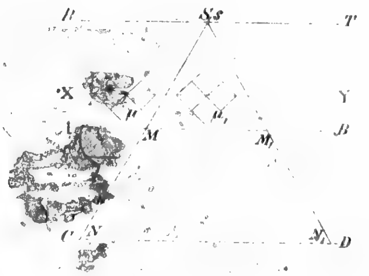


Fig. 126

$$x = d \frac{MM_1}{NN_1 - MM_1}$$

The two lines and the points NN_1 must be taken as far apart as the instrument will allow.

Having the distance of the front line AB, other front lines at fixed distances from the foot of the station are permanently marked on the drawing board.

The ground plan can now be placed on the board by putting the trace of the principal plane in coincidence with the line drawn on the board and placing a front line of which the distance is known upon the corresponding one of the drawing board.

93. TO FIND THE DISTANCE BETWEEN THE TWO SLIDES.

The distance between the two slides is equal to the distance line of the perspective: it must be indicated by a scale on the edge of the drawing board. In order to locate the zero of the graduation the precise distance has to be determined in one position of the instrument.

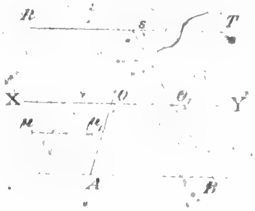


Fig.127

Draw a front line AB; Fig.127. Put the tracer first in A and then in B, marking in each case the positions μ and μ_1 occupied by the pencil. Let d be the distance between the slides and m the distance from s to AB: we have:

$$\frac{d}{m}=\frac{OO_1}{AB}$$

AB being a front line, its perspective $\mu\mu_1$ is parallel to XY and equal to OO_1, consequently:

$$d=m\frac{\mu\mu_1}{AB}$$

The three lengths which form the second term, can be measured and the value of d calculated.

94. TO DRAW THE GRADUATION FOR THE HEIGHT OF THE STATION. The height of the station is represented on the instrument by sS_1 (Fig. 124). It is necessary to determine this distance for one position of s and S_1, in order to draw the graduation for setting S_1 to any required height.

Place the tracer M on the trace of the principal plane at a distance Ms equal to one and a half times $s0$ and note the place occupied by the pencil μ. In this position of the instrument we have

$$0\mu_1 = \frac{1}{3}\, s\, S_1$$

Then place the tracer M still on the trace of the principal plane but at a distance Ms equal to three times $s0$: we have

$$0\mu_1 = \frac{2}{3}\, s\, S_1$$

The change in the value of $0\mu_1$ is thus equal to one third of the height of the station; but this change is represented by the displacement of the pencil μ, which can be measured with a scale. Three times this displacement is the height of the station.

The tracer, instead of being placed at the distances from the foot of the station given above, may be set at any distance which may be convenient; the fraction of the height of the station obtained will be different, but the process is the same.

95. TO DRAW THE HORIZON, GROUND AND PRINCIPAL LINES ON THE PERSPECTIVE. The principal line is the perspective of the trace of the principal plane on the ground plane. The latter has been marked on the board (§ 89): following it with the tracer, the pencil describes the principal line.

The ground line cannot be drawn directly, because the tracer would have to be carried along the front slide and the construction of the instrument does not permit it. The difficulty is overcome by drawing the perspective of a front line between the ground and horizon line.

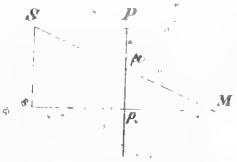


Fig. 128

Let Fig. 128 represent the principal plane and M the trace of a front line of the ground plane at a distance sM from the foot of the station equal to twice the distance line. The similar triangles SsM, μpM give

$$\mu p = Ss\,\frac{pM}{sM} = \frac{1}{2}Ss$$

Ss is equal to the height Pp of the principal point above the ground line, therefore,

$$\mu p = \frac{1}{2}Pp$$

Following then with the tracer the front line drawn on the board at a distance from the station equal to twice the distance line, the pencil will describe a horizontal line midway between the ground and horizon lines. One half the height of the station is now measured on each side of the line so obtained and parallels drawn to it. The line nearest to the front slide is the ground line, the other one is the horizon line.

Processes similar to those given for the perspectograph can be employed for the other perspective instruments.

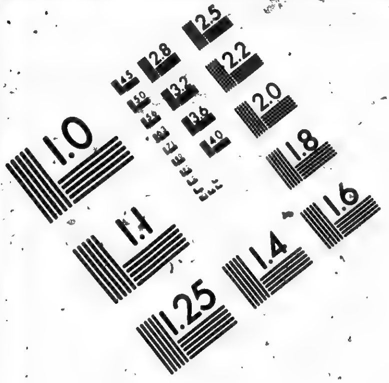
With the camera lucida, however, particular attention must be paid to the fact that the station is not the same for the direct and reflected rays § (88).

Although the fact has no importance when drawing the perspective of remote objects, it would if not allowed for, cause considerable errors when the subject and the perspective are both close to the station.

96. CENTROLINEAD. In addition to the instruments already described, others have been devised merely to facilitate the construction of perspectives. They are not properly speaking, perspective instruments, since they do not enable the draughtsman to draw the perspective directly.

The vanishing point of a line nearly parallel to the picture plane being at a great distance from the principal point, may fall outside of the paper, in which case special constructions are necessary to draw a line which, if produced, would pass through the point.

The "centrolinead" permits to draw a line vanishing at any point, no matter how far from the principal point. It consists of a straight edge, (Fig.129) with two arms whose inclination to the straight edge can be varied at pleasure. Two studs six or eight inches apart, are fixed to the edge of the drawing board. The arms of the centrolinead being placed in



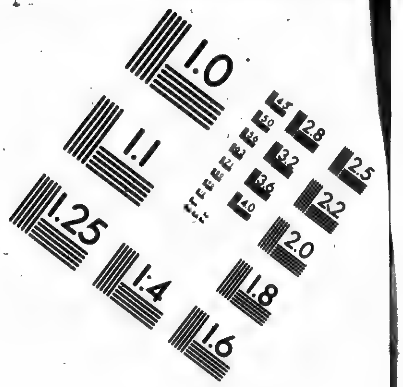
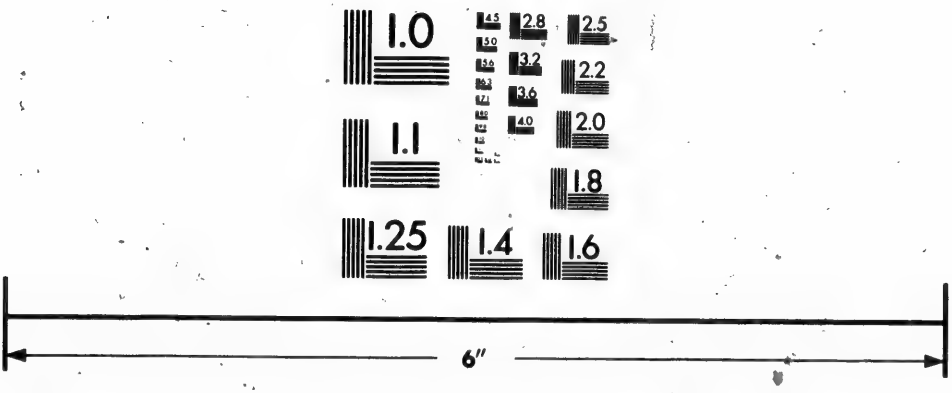
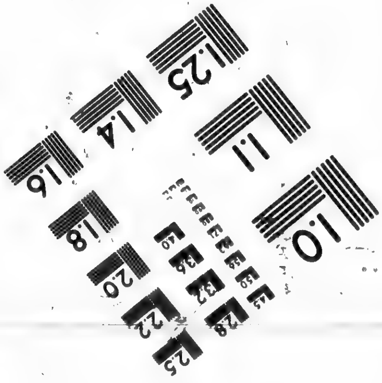


IMAGE EVALUATION TEST TARGET (MT-3)





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contact with the studs the various directions of the straight edge intersect at a common point.

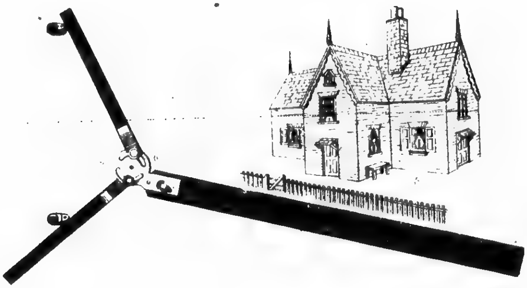

Fig.129

Let OC, Fig.130, be the straight edge, OA and OB the arms, A and B the studs. Through A, O and B, pass a circle: the arms being fixed in a certain position, the angle O is constant and is bound to move on the circumference of the circle whenever the position of the centrolinead is changed, as from OC to O'C'.

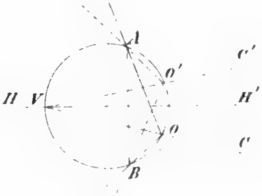

Fig.130

Produce OC and O'C' to

their second intersection with the circumference: they must cut it at the same point V, because the angle AOV being invariable must always subtend the same arc AV, no matter on what point of the circumference the apex O may be. Consequently the straight edge will draw all the lines vanishing at V.

The centrolinead is employed only for horizontal lines, whose vanishing point is on the horizon line. The studs A and B are placed on the same perpendicular to the horizon line and at equal distances from it; it follows that the horizon line HH′ is a diameter of the circle, and,

$$VA = VB$$

The arms are equally inclined to the straight edge. The line OC, bisecting the angle AOB must pass through V which is the middle of the subtended arc BVA.

The distance of the vanishing point, V, can be varied either by changing the positions of the studs or the inclination of the arms. Increasing the distance AB between the studs, the size of the circle is increased in the same proportion and V moved to the left.

It is not usual to disturb the studs, the changes in the distance of the vanishing point being obtained by altering the inclination of the arms of the centrolinead. Were the arms perpendicular to the straight edge, the vanishing point would be at infinitum and the instrument would describe parallels to the horizon line.

Closing the arms gradually, V comes nearer to AB; when AOB becomes a right angle, the intersection of AB and HH′ being the centre of the circle the distance of V from AB is one half of the latter.

Closing again the arms, V continues to move towards AB without ever reaching it.

In reality, the studs are not mathematical points, but cylinders: the direction of the straight edge is, however, the same as if the arms rested against the axes of the cylinders.

The direction of the vanishing point may be given by a line of the ground plan or by a line of the perspective. In either case, the arms of the centrolinead have to be set to correspond to the vanishing point.

In the first case, revolve the picture plane on the horizon plane around the horizon line as an axis.

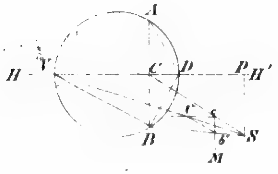

Fig. 131

The station comes in S, Fig. 131, SP is the distance line, A and B the two studs. Let SV be the direction, of the given line on the ground plan, V is its vanishing point. Through A, B and V, pass a circle; the centrolinead should be set so that the straight edge being on DH', the arms should be on DA and DB.

Join VB; the angle VDB, inclination of the arm on the straight edge, is equal to the angle VBA, because they subtend equal arcs. Join CS and BS, and draw Mc and cv parallel to AB and HH'; join bv. By reason of the similarity of triangles, bv is parallel to VB and the angle

$$vbc = VBC$$

Therefore place the straight edge on MB, the axis of rotation on b, and adjust the left arm of the centrolinead to coincide with bv. The other arm may be set by placing the straight edge on *vb*, the axis on *b* and adjusting the arm to coincide with *b*M, or better by placing the straight edge on the horizon line, the arm already adjusted in contact with the stud B and moving the other arm until it comes in contact with the stud A.

The lines BS, CS, M*c* and *cv*, are drawn once for all, *bv* need not be drawn, so that the only line to be marked is S*v*, direction of the given line on the ground plan.

When the given line belongs to the perspective the construction must be slightly modified. VE, Fig. 132, being the given line, take any point, F, on the horizon line, join FE and FB, and draw *c*M parallel to AB. Through *e*, draw *ev* parallel to EV and join *vb*. On account of the similarity of triangles, *vb* is parallel to VB and the angle *vbc* is equal to VBA, inclination of the arm on the straight edge of the centrolinead.

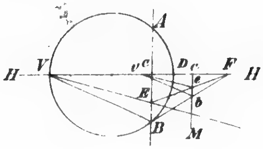

Fig. 132

FB and *c*M are drawn once for all but FE and *ve* have to be marked for every given line: that is, two lines instead of one by the former construction.

Centrolineads are usually sold in pairs, one to work on the right of the principal point and one for the left.

97. PERSPECTOMETER. In § 65, a method has been given for transferring a figure from the perspective to the ground plane by means of squares formed of lines parallel and perpendicular to the ground line. The "perspectometer" has for object to dispense with the construction of the squares' perspective.

On a piece of transparent material, glass, horn or celluloid, draw two parallel lines AB, DD′, (Fig. 133) and a common perpendicular P*p*. Take DP, PD′, *p*A and *p*B equal to the distance line and from *p*, lay on AB equal distances

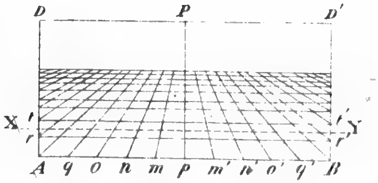

Fig. 133

pm, mn, pm'........Join to P the points of division m, n, m', n'.......... part of those lines intersect AD and BD′ at r, t, r', t',........ The corresponding points are joined together by lines which are parallel to AB and DD′.

The instrument is now placed on a perspective, with P on the principal point and DD′ on the horizon line. The ground line will fall in XY, for instance: it is divided into equal parts by the lines converging in P and the figure of the perspectometer is the perspective of a network of squares in the ground plane, having the equal parts of XY as sides. By referring to § 65, it will be seen that the construction is precisely the same in both cases.

This instrument is useful for restoring from the perspective a figure of the ground plan. By placing it on the perspective the squares covering the figure are at once apparent and only those required are drawn on the ground plan.

The side of the squares is equal to the length intercepted on the ground line between two of the converging lines: this distance is laid on the ground plan from the trace of the principal plane and parallels drawn to the trace through the points of division.

The front line nearest to the ground line is laid on the ground plan either by estimating its distance from the ground line or by constructing it. The estimation is made by noting the fraction of a square's side which represents the distance from the ground line.

Figures in planes inclined to the horizon but perpendicular to the picture plane are transferred to the ground plan by placing the centre of the perspectometer on the principal point and the parallel lines in the direction of the trace of the inclined plane on the picture plane. The trapezoids of the instrument are the perspectives of squares in the inclined plane, which squares are projected as rectangles on the ground plane (§ 77). The longest sides of the rectangles are perpendicular to the picture plane and equal to the length intercepted between two converging lines of the instrument on the trace of the inclined plane. The shortest sides are the projections on the ground line of these intercepted lengths.

The rectangles are constructed on the ground plan and the transfer made from the perspective as in the preceding case.

When the plane containing the figure is inclined to the picture and ground planes, the principal point must first be displaced on the principal line so as to project the figure on a plane perpendicular to the picture plane and having the same trace as the given plane, (§ 80): the problem is now the same as the last one.

The perspectometer can only serve for perspectives having the same distance line, such as photographs of distant objects taken with the same lens; every distance line requires a new instrument. The width Pp should be equal to the height of the horizon line above the foot of the picture; the length DD' need not be larger than the picture, the distance points being placed on the figure merely for the purpose of demonstration.

The length of the equal parts of AB should be such that the divisions of the lowest ground line employed be not too large for the degree of accuracy required. These divisions are the sides of the squares or rectangles of the ground plan and the larger their size, the less accurate will the transfer of the figure be.

The instrument can be made by drawing it on a large scale on paper, and taking a reduced negative from which a positive is made on a transparency plate. The transparency is bleached in bichloride of mercury and varnished: the lines originally black, are now white on clear glass.

98. DRAWING THE GROUND PLAN WITH THE CAMERA LUCIDA. The distinction between the picture and ground planes is purely conventional; the picture plane may be taken for ground plane and the ground plane for picture plane. If α be the perspective of the figure A of the ground plane, inversely A is the perspective of the figure α of the picture plane. Consequently any perspective instrument can be employed to draw the ground plan from the perspective, by a change in the setting of the instrument, the distance line being now what was formerly the height of the station and

inversely the new height of the station being the former distance line.

With the camera lucida, the prism must be fixed permanently so that the height of the virtual station $S'P$, Fig. 134, above the plane on which the perspective is placed, be equal to the distance line of the perspective (§ 86). As long as the latter does not change, the prism must remain in the same position. The perspective is placed in $HH'XY$, in such a manner that the line $S'P$ joining the virtual station to the principal point of the perspective be perpendicular to the plane of the table or drawing board.

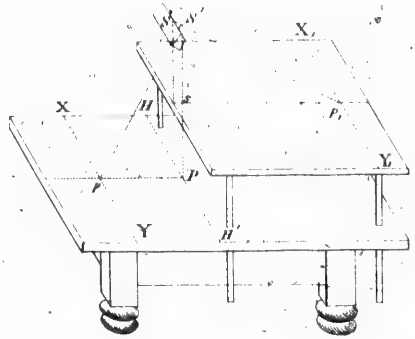

Fig. 134

The ground plan is on a platform sX_1 Y_1, parallel to the plane of the table, and which can be moved up and down. It must be so placed that the perpendicular

Ss from the real station to the platform, be equal to the height of the station: s is the foot of the station and the ground line is somewhere in X, Y.

The eye looks by reflection at the perspective and sees directly the pencil on the platform.

The determination of the constants is made by methods similar to those of the perspectograph (90, 91, 92, 93).

The camera lucida permits to draw the ground plan, either within or beyond the ground line: it is a very serious advantage. Against it is the want of precision due to the size of the notch in the mounting of the prism and the resulting displacements of the eye.

99. DRAWING THE GROUND PLAN WITH THE CAMERA OBSCURA. In order to transfer a figure from the perspective to the ground plane, the camera obscura would have to be set as in Fig. 135, the perspective HHXY being placed vertical and the ground plan horizontal in sAD. The distance from the optical center of the lens, S, to the perspective should be equal to the distance line and the height of S above the plane sAD, equal to

the height of the station. The difficulty is that only one line would be in proper focus, the other portions of the image being more or less blurred. It might be possible, by using a lens of proper focus and a pin hole diaphragm, to obtain fair definition within a limited space ABCD, but it is doubtful whether the process would prove practical.

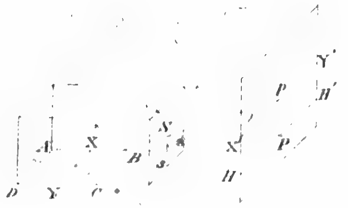

Fig.135

100. DRAWING THE GROUND PLAN WITH THE PERSPECTOGRAPH. In order to draw the ground plan from the perspective with the perspectograph, the distance between RT and XY (Fig.124) must be equal to the height of the station, and *sS* to the distance line. The principal point of the perspective must be placed in *s*, the horizon line under RT and the ground line under XY. The instrument thus arranged would not work with the perspectives used in surveying, it could not even be

set, the obliquity of the arms being too great. It may, however, be employed to transfer a figure of the perspective to other planes than the ground plane, as for instance to obtain the elevation of a building from the perspective of the facade; the method is fully described in Ritter's pamphlet (1).

In other cases, and particularly for topographical purposes, the pencil must be placed in M and the tracer in μ. The perspective is placed under μ, with the ground and principal lines on the lines previously marked on the drawing board (§ 92). Taking M with the hand, the arms are moved so as to follow the lines of the perspective with the tracer μ. The operation being precisely the same as for drawing the perspective from the ground plan (§ 88) it is evident that the pencil in M will now reproduce the ground plan. The use of the instrument in this manner is at first a little difficult, owing to the point M being guided by the hand while the perspective has to be traced with μ, whose motion is entirely different; some

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(1) See note, page 158.

practice is wanted before being able to handle it successfully.

A certain amount of dead motion is inevitable in an instrument of this kind, particularly when changing the direction of the motion perpendicular to the slides XY and RT. In order to avoid the errors which this would introduce, the pencil M should always be moved in the same direction, away from XY for instance. When the draughtsman comes to a part of a line or curve which is directed towards XY, he should lift the pencil, push M back to the other end of the curve and trace it in the opposite direction.

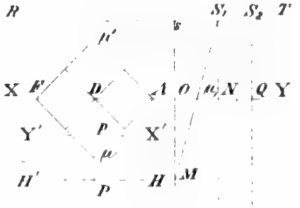


Fig.136

The position of the horizon line HH′ of the perspective, Fig.136, varies every time the distance from s to S_1 is changed, for it corresponds to the tracer μ when M is at infinitum and the two arms parallel: μ_1 would then be in N. Now change the height of the station from S_1 to S_2 ;N comes in Q carrying with it the joint A of the pantograph system to which it is rigidly fixed,DA is increased by NQ and μ moves down the same quantity. So the horizon line is displaced towards the front of the drawing board the same distance as the station is moved up. The ground line is not affected. The instrument is provided with means of adjustment for the distance from A to μ_1 ,these two points being connected by an iron rod sliding in a ring at A,to which it can be fixed by a clamping screw. A graduation placed along side the rod permits to add to Aμ the increase in the height of the station,in which case the horizon line of the perspective does not move and both the perspective and the ground plan occupy invariable positions on the drawing board,no matter what ground plane may be used.

When the distance line is changed,as for figures in inclined planes,the simplest manner to place the perspective on the board is to put the pencil in M on the trace of the principal plane (Fig.136),MO being equal to Os ;then slide the perspective under the tracer until the latter is on a point midway between the ground and horizon lines. The principal line of the perspective must of course,coincide with the line previously drawn on the drawing board.

101. CHANGE OF SCALE. The perspectograph set as in Fig.124,will only work on points beyond the picture plane. It is possible,by placing the slide XY on the other side of RT,to reach within the picture plane,but the instrument does not work well. It is preferable to use it as set in Fig.124 and to resort to a change of scale when figures within the picture plane are to be operated upon.

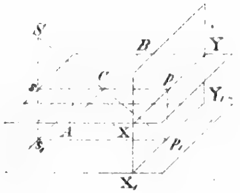


Fig.137

Let sXY and XYX_1Y_1 Fig.137,be the ground and picture planes. Take a new ground plane, $s_1X_1Y_1$, at a distance ss_1 below the other plane equal to the height of the station Ss ,the figures obtained on the new plane from the perspective,will be on a scale double of the former scale,(§ 53). The new ground line X_1Y_1 corresponds to the front line AB of the plane sXY,midway between the foot of the station, *s*, and the picture plane, so that it will now be possible to work with the perspectograph on the part of the ground plane comprised between AB and XY, but the result has to be reduced to half size.

By doubling again the scale, one half of the space between AB and *s*, will be covered; the result must be reduced four times.

In practice the draughtsman would commence by working on the figures beyond the picture plane. After transferring them to the ground plan, he would move S, , Fig. 124 so as to double *s*S, and draw a new ground line on the perspective, by doubling the distance of the first one from the principal point. He would then place the perspective in proper position for the new ground line and continue to operate the instrument as before.

The restored ground plan is drawn on cross section paper, having squares of four sizes, distinguished by lines of different intensities. The largest squares are divided into four smaller ones and the latter are also divided into four. The sides of the squares are even divisions of the scale. When the scale is increased two or four times, the reduction of the figures is made at sight by transferring the figures from one set of squares to the other. Front lines have been marked on the drawing board at even distances from the foot of the station. The cross section paper is pinned to the board with some of its lines upon the front lines of the board and a perpendicular to the former on the trace of the principal plane.

The distance from the foot of the station is marked on one of the front lines of the paper forming the sides of the squares: this distance, with the trace of the principal plane permits to transfer to the general ground plan the portion of it which has been obtained by the perspectograph.

When the scale is changed, the distance of the front lines should be modified accordingly. Thus if the scale be doubled, the front line marked 10 on the drawing board will correspond to a real distance of 5 and should be so marked on the paper.

CHAPTER IV

FIELD INSTRUMENTS

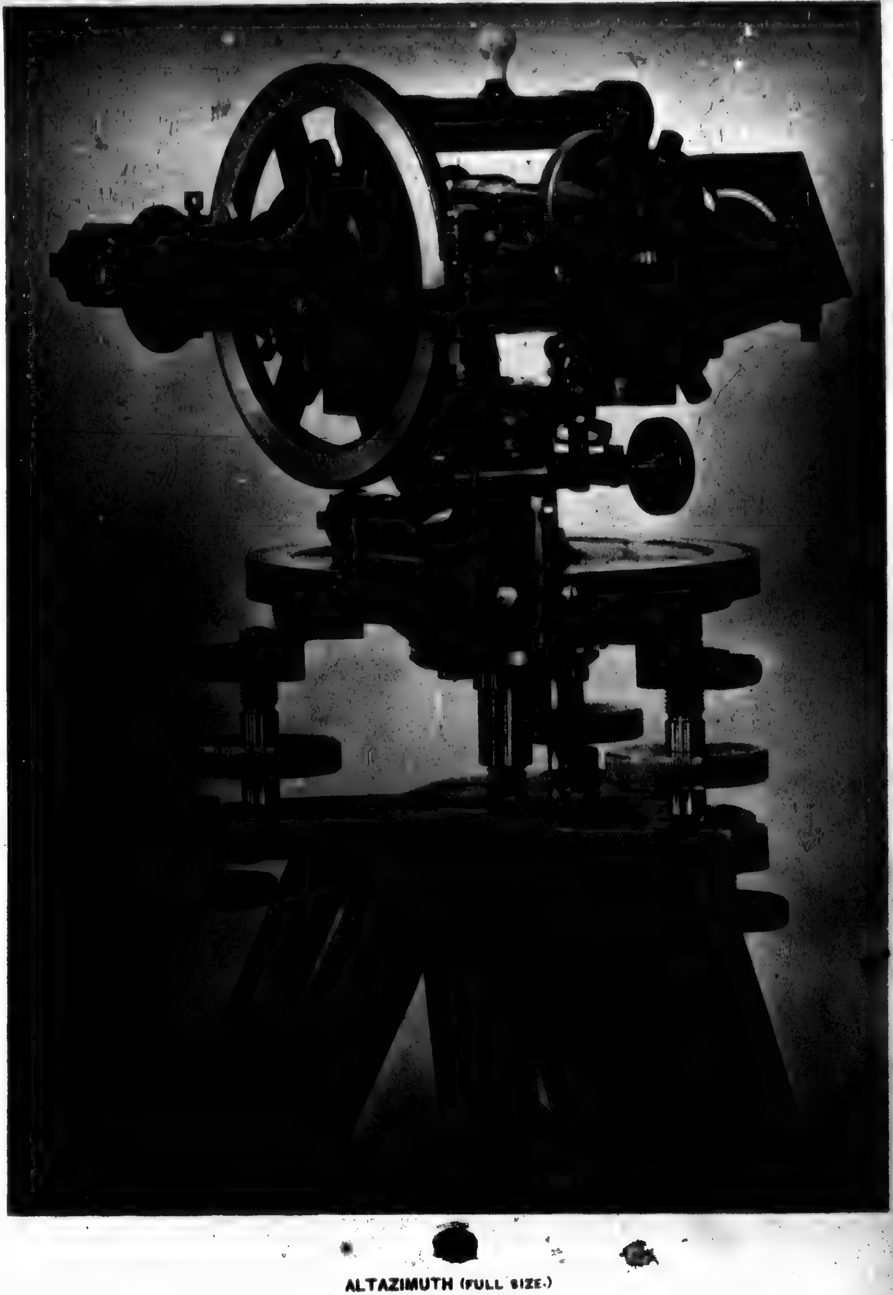
102. **ALTAZIMUTH.** The instruments used in the field on Canadian Photographic surveys consists of an altazimuth and a camera.

The altazimuth is represented in full size on Plate IV: it is made by Steinheil Söhne of Munich. Its peculiarity is the telescope, which is horizontal and has a reflecting prism in front of the object glass. This disposition, which has been adopted by several makers for small instruments, presents two important advantages: the apparatus is very compact and the great distance between the bearings of the telescope is favourable to the precision of the results.

The weight of the instrument is 2 lbs 10 oz;with box complete 5½ lbs. The horizontal circle has a diameter of 8 centimeters and the vertical circle 7 centimeters,each being read to one minute by two verniers. The vertical circle is fixed to the telescope with which it revolves.

The telescope rests in the Y's on four agate points,set on the top of brass screws with stop nuts, which permit to adjust the axis. The object glass is a "triple" one,being composed of a crown lens between two flints: it has an aperture of 20.3 millimeters and a focus of 8.1 centimeters. The magnifying power is about 7:the definition is remarkably good.

The right angle prism in front of the object glass,reflects the rays of light by the hypothenuse face,which is inclined at 45° on the two other faces. The hypothenuse is of course polished glass,with the exception of a ring not covered by the mounting,where the glass is ground. By placing a lamp in the direction of the telescope's axis,the diffused light admitted through the ring gives a very good illumination for the wires. The slow motion to the vertical



ALTAZIMUTH (FULL SIZE.)

circle and telescope is given by an endless screw, which can be lifted out of bearing for rapid motions. The telescope may be revolved end for end in its Y's: the verniers of the vertical circle not being attached to the circle, there are two pairs of them, one for each side. There are two levels, one for the horizontal circle and a striding level. The usual coloured and reading glasses are provided.

The defect of this instrument, which is common to nearly all those of the same class, consists in its numerous adjustments. They are particularly objectionable for surveys in a mountainous country where occasional rough usage is unavoidable. It is much to be desired that the makers should supply small instruments without any adjustments; they could be made sufficiently accurate for ordinary work, and for precise operations, a surveyor does not trust to his adjustments in any case, but determines the errors of the instrument and applies the corresponding corrections to the observations.

103. ADJUSTMENTS. This altazimuth being of an unusual pattern, it may not be out of place to describe

its adjustments.

The verniers of the vertical circle can be moved up and down by means of screws: they must first be placed in the middle of their course and the telescope raised or lowered by its supporting screws until the vertical circle reads at the same time 0° and 180°. The same supporting screws will displace the telescope to the right or left and care must be taken that the verniers impinge equally on the graduation.

The level of the horizontal circle is adjusted as in other instruments.

The next point is to ascertain whether the two collars of the telescope have the same diameter and, if not, to determine the difference.

Fig. 138

With the striding level, level the upper face CD of the telescope (Fig. 138). Should the two collars not be equal, the lower face AB will be inclined to the horizon and the inclination of the telescope's axis will be one half that of AB.

Reverse the telescope in its Y's:the upper face takes the position CD',of which the inclination is twice that of AB. The striding level being set on CD' will be out by a certain number of divisions which is the measure of the inclination of CD'. One half that quantity is the angle of the two faces of the telescope or more precisely,the angle of the cone determined by the two collars.

Then level the vertical axis of the instrument by the ordinary methods,and with the supporting screws of the telescope,move the latter until the striding level indicates for the upper face CD,an inclination equal to the angle of the two faces:the lower face AB is then horizontal. The inclination of the optical axis is one half that of the upper face: it will be the same when the telescope is reversed, but in the opposite direction. In adjusting the supporting screws,it is necessary to pay particular attention to the verniers and to see that they overlap equally on the graduation of the circle.

The wires are now placed in focus for distant objects and the draw tube of the telescope fixed permanently by the screw provided for the purpose.

The prism is then taken off, the telescope set on a distant object and turned round in its bearings.

The wires must be adjusted until their intersection remains constantly on the same point while the telescope effects a complete revolution. The optical axis then coincides with the axis of figure of the telescope and the point sighted upon is on the same level as the instrument, if the collars of the telescope have the same diameter; if not, the altitude of the point is equal to the inclination of the optical axis, which has been measured.

Now replace the prism in front of the object glass, set the telescope on a distant point and reverse it in its bearings. The vertical wire will probably not cover again the same point: correct one half of the difference by the adjusting screws of the prism and repeat the operation until the point is bisected by the vertical wire both before and after reversal. The line of sight is then perpendicular to the axis of the telescope.

The next adjustment is that of the verniers. Set the telescope on the point used in adjusting the wires and which is in the horizon of the instrument. The readings of the verniers should be 0° and 180°: move them until they read accordingly and turn the instrument 180° in azimuth. Setting again the telescope on the same point, the readings should be 180° and 0° if the circle were exactly centered on the axis of figure of the telescope. This being seldom the case, the line 0° 180° which before reversal coincides with the line AB, Fig. 139, joining the 0's of the verniers, will be in $A'B'$ after reversal. The vernier A will read for instance 179° 50' and B, 0° 10'. The verniers must be moved over half the difference: in the present case they should be made to read 179° 55' and 0° 05'. After turning the instrument 180° in azimuth, they will show the same figures, so that the altitude of a point will be given

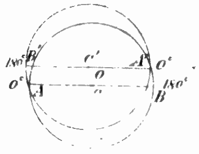


Fig. 139

at once by taking the mean of the two vernier readings in one position of the telescope.

The point sighted upon was taken in the horizon of the instrument:any other point may be employed,its correct altitude being determined by the mean of the two altitudes obtained by observing in one position of the instrument, turning it 180° in azimuth and observing again. Instead of setting the verniers to 0° and 180° in the first instance,they are set to the correct altitude and the adjustment is completed in the same manner as before.

104. TRIPOD. The tripod is a sliding one:it draws to a length of 38 inches which experience has shown to be sufficient in a mountainous country:it weighs 3½ lbs and when packed is 20½ inches long. It was adopted at first in the absence of anything better,the intention being to replace it as soon as a proper one could be procured. Contrary to expectations,it has proved so steady that it was decided to keep it.

The tripod serves for the altazimuth and the camera,both instruments being secured by the same screw with spiral spring,which forms part of the tripod head. Three small levelling screws passing through the head permit to level the camera.

Both instruments being very light, steadiness is secured by a net between the tripod legs, on which a heavy stone is placed. With this device, not only are the observations and photographs better, but there is no risk of the instrument being blown away during one of those wind blasts so frequent in the mountains.

105. USE OF ALTAZIMUTH. The instrument being set up and levelled, the surveyor sits opposite one of the verniers of the horizontal circle, his assistant sitting at the other vernier. Turning the prism of the telescope to the right, he commences by observing the points immediately behind him. For doing so he bends to the left and looks through the telescope whose axis is directed towards the right. Without leaving his seat he reads the two verniers of the vertical circle, the vernier of the horizontal circle and enters the readings in his book. The assistant reads his vernier and notes it in his own book.

The same operation is repeated for every point on the right of the surveyor, until those immediately in front are reached. For observing them, the surveyor bends to the right and looks towards his left.

The prism of the telescope is now turned towards the left and all the points on that side observed in the same manner as those on the right. The surveyor thus completes a series of readings around the whole horizon, 180° being with prism to right and 180° with prism to left. The instrument is then turned 180° in azimuth, the eye piece of the telescope being on the side of the assistant, who observes to the right and left as did the surveyor before and enters in his book the readings of the vertical circle and of his vernier of the horizontal circle. The surveyor merely takes the reading of the vernier of the horizontal circle in front of him.

Two complete series of readings around the horizon have now been obtained, one with prism to right and one with prism to left and during the whole time, neither the surveyor nor the assistant has left his seat. Should more precise observations be wanted, the same operations may be repeated after reversing

PLATE IV

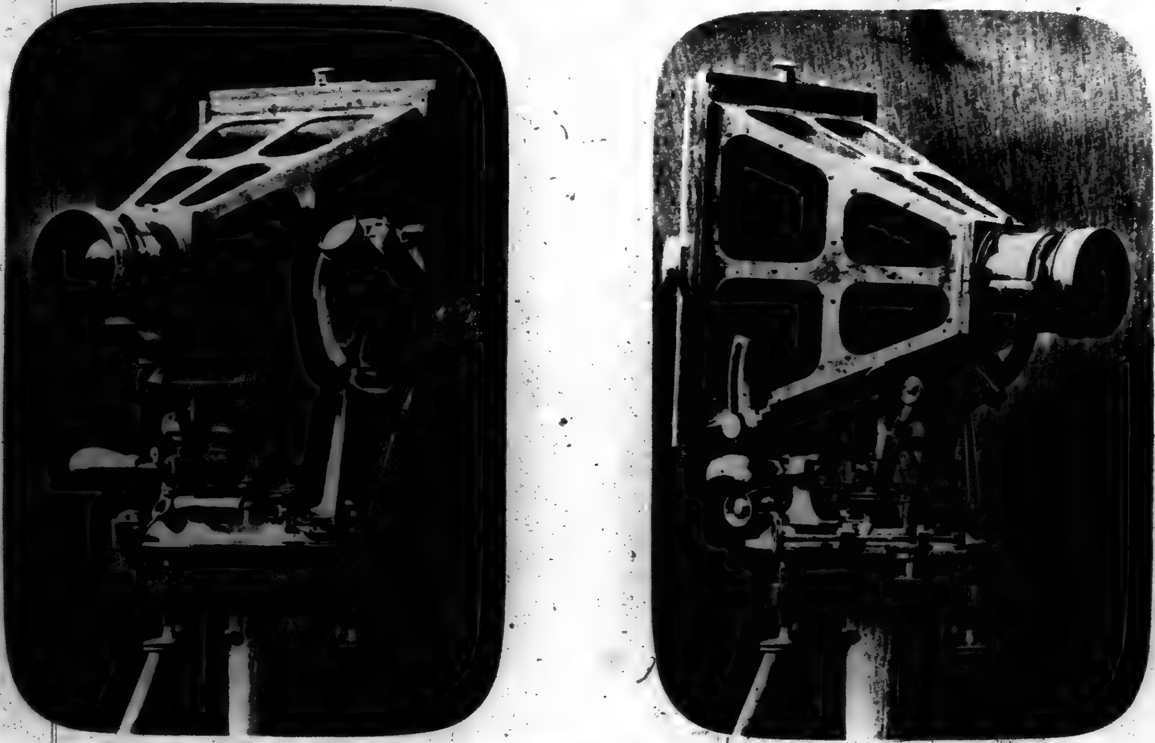


PHOTO-TOPOGRAPHICAL APPARATUS
OF THE GEOGRAPHICAL MILITARY INSTITUTE OF ITALY.

the telescope in its Y's.

After completing the observations, the surveyor and his assistant compare notes: every point having been observed and every reading taken, independently by surveyor and assistant, any error is discovered at once and rectified on the spot.

106. CAMERAS. A large number of cameras adapted to surveying, have been constructed; although some of them are very ingenious, they have not come into general use. They are fully described by Lieut. Reed in "Photography applied to Surveying".

The instrument employed by the General Staff of the Italian Army (Plate IV) is an altazimuth and camera combined, the object being to measure with the telescope and graduated circles not only the angles required in ordinary surveying, but also the direction of the principal point of the perspective. The vertical circle and telescope are on the side of the instrument, and the camera which is of metal and has the form of a pyramid, is in the center above the horizontal circle. It can be detached from the altazimuth when not in use.

The German pattern, Fig.140, is due to Dr.Meydenbauer: it is also a metal camera, in the form of a pyramid, set on a graduated circle with levelling screws. The views embrace an angle a little over 60° and the mode of using it consists in turning the camera precisely 60° in azimuth, after each view, by means of the graduation, so as to obtain a complete panorama with six views. It is intended to be employed alone, all angles required for plotting the plan being taken from the photographs.

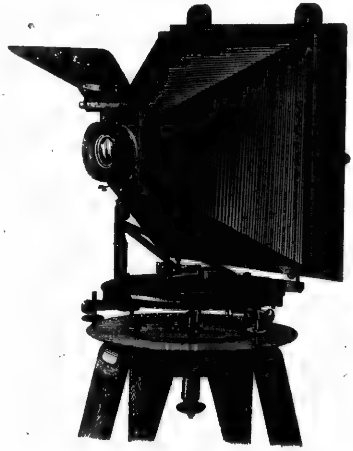


Fig.140

This instrument was devised and employed principally for making plans of buildings, for which it may be well adapted. For topographical purposes, a separate instrument for measuring angles is a necessity, so Dr.Meydenbauer's camera could not be used alone, as intended. With the two instruments, the graduated circle of the camera becomes useless and only adds to the bulk and weight. Great superiority is claimed for the metal camera over a wooden one, on account of its greater precision. Without entering into a discussion on the comparative merits of wood and metal for instruments of precision, it may be stated that a camera of seasoned mahogany, well bound with brass, will for topographical purposes, give all the accuracy that photography applied in a practical way, is able to give.

The possibility of taking a complete panorama with six views does not prevent any important advantage. On the Canadian Surveys, seldom more than two or three views, together embracing an angle of 120° to 180°, are taken from one station, or are found to be useful.

A surveying camera merely requires to be simple, strong, and without any adjustments liable to get out of order, except the means of setting the sensitized plate vertical, which is a necessary condition to obtain correct perspectives. Any camera fulfilling these requirements is adapted to surveying.

107. **LENSES.** The choice of the lens is of great importance;no satisfactory results can be expected with an imperfect instrument.

There are several kinds of lenses employed for landscape photography;some consist of a single achromatic lens and others of two combinations,either similar or not.

The single or landscape lens does not give correct perspectives:if,for instance,parallel and perpendicular lines be ruled on a sheet of paper so as to divide it into squares and a photograph taken with the paper parallel to the sensitized plate,the image does not consist of squares,but is a figure either spindle or barrel-shaped,according as the diaphragm is behind or in front of the lens. In photographing a subject with such a lens, the only vertical lines represented by straight lines, are those in the principal plane;the perspectives of other verticals are curves more and more inclined as

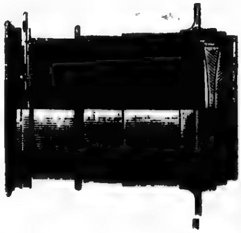


Fig.141.

the distance from the principal point increases. An essential condition of a correct perspective is that all vertical lines be represented in the perspective by parallels to the principal line, therefore the photographs taken with an ordinary landscape lens are not suited to surveying.

Fig. 141 represents an ordinary landscape lens with rotating diaphragm.

Mr. Dallmeyer has brought out lately a landscape lens which he calls "rectilinear". It is probable, however, that the distortion is merely reduced without having disappeared entirely. The landscape lens has over all others the great advantage of a smaller number of reflecting surfaces and minimum thickness of glass: less light being absorbed by the glass and lost by reflection, the images are so much brighter. It would be the best lens for surveying purposes, if it could be made absolutely rectilinear but that does not seem possible.

The lens most generally employed is a combination of two similar lenses between which the diaphragm is placed. It has received different names from the makers, such as the "Rapid Rectilinear" of Dallmeyer, "Rapide Rectilinéaire" or "Aplanétique" of Français and Hermagis, "Aplanat" of Steinheil, "Euryscope" etc., but all these lenses are constructed on the same plan, although there are differences in the curves and kinds of glass employed.

To understand how the distortion is corrected, the double combination may be supposed to consist of two single landscape lenses, between which the diaphragm is placed. The front lens, having the diaphragm behind causes a spindle-shaped distortion which is counterbalanced by the barrel-shaped distortion due to the back lens having its disphragm in front. This must be taken only as a rough explanation of the lens; a double combination does not consist of two landscape lenses, and the correction of the distortion is obtained by calculating the curvature to be given to the lenses in order to produce the effect desired.

In the "rapid" variety of this form, the distance between the two combinations is about equal to their diameter, the diaphragm being placed in the middle.

This arrangement permits the use of a large aperture, the lens being what is called a "rapid" one. The definition, good at the centre, decreases rapidly towards the margin of the plate and can only be improved by the use of small diaphragms.

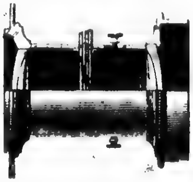


Fig.142

Any point near the centre of the plate receives the light coming through the whole of the lens' opening, but owing to the great distance between the two combinations, part of the light is intercepted by the mounting when the point is near the edge of the plate. The result is a rapid decrease in the brightness of the image from the centre to the edge, which can be improved only by the use of smaller stops.

Fig.142 represents Henry Hermagis' "Rapide Rectilineaire" lens with attached diaphragms.

In other lenses constructed on the same plan, the distance between the two combinations is reduced more or less: they are designated under the general name of "Wide Angle" lenses. The diaphragms being

close to the lenses, their diameter has to be reduced, so this form is slower than the rapid rectilinear. On the other hand the diminution of the distance between the two combinations causes a more uniform distribution of the light over the plate, the brightness of the image not changing so rapidly as in the rapid rectilinear. The definition keeps good at a greater distance from the centre: if a rapid rectilinear and a wide angle be stopped down to the same aperture, it will be found that the latter will give good definition over a larger area than the former; it will cover a larger plate, hence the name of wide angle.

Fig. 143

Fig. 143 represents Dallmeyer's wide angle lens with rotating diaphragm.

Comparing the defects and advantages of the various lenses, the wide angle is indicated as the lens for surveying purposes. Rapidity is unnecessary but uniform illumination and definition, in which the wide angle excels, are important. A well constructed wide angle lens is practically free from distortion. A simple way of testing it consists in dividing a sheet of paper into squares by parallel and perpendicular lines. Place it parallel to the ground glass of the camera and take a photograph: the result will show at once whether there is distortion. With a good lens no error should be seen without a microscope: the measurements and constructions on the photographs which will be described later on being all made with the naked eye, it follows that the distortions due to such a lens may be neglected.

The terms of "back focus" and "equivalent focus" are often found in manufacturers' catalogues and in books relating to photography. The "back focus" is the distance from the flange to the focal plane: it has no particular meaning beyond expressing the length to which a camera must extend to be used with a certain lens.

The "equivalent focus" is the focal length of the single lens which would give the same image as the lens referred to. It is seldom given accurately in the catalogues. To find it, draw a geometrical figure such as a square or triangle, on a plane parallel to the ground glass, focus and move the camera until the image seen on the glass is equal to the figure itself: mark the position on the bed of the camera. Then focus on very distant objects and mark the new position; the distance between the two marks on the camera bed is the "equivalent focal length" of the lens.

In elementary text books on optical matters, several processes are given for measuring the focal length, which rest on the assumption of an optical centre through which all rays of light pass without being refracted. For instance it is suggested to measure the distance between a figure and its image when both are equal and to divide by four: this method depends on the hypothesis of the optical centre. It must not be forgotten that there is no such point, that it is merely a convenient hypothesis for facilitating demonstrations or explanations and that any method depending upon it gives only an approximate result. In every lens or combination of lenses, there are two points called "nodal points" such that any incident ray of light passing through one of the nodal points will emerge from the lens in a direction parallel to the incident ray and passing through the other nodal point.

The equivalent focal length is the distance between the second nodal point and the focal plane.

The brightness of the image varies proportionally to the size of the lens or to the square of its diameter. The larger the aperture, the more light is admitted. It varies also inversely as the square of the focal length: thus if the focal length be doubled, the aperture remaining the same, an equal quantity of light is admitted but is distributed over an area four times larger, the brightness of the image being reduced in the same proportion.

Representing the aperture by a and the focus by f, the brightness of the image is therefore proportional to

$$\frac{a^2}{f^2}$$

This fraction is the measure of the rapidity of a lens or of its capacity to produce a certain luminous effect such as the impression of a photographic plate. The aperture referred to is the effective aperture of the lens which may be much smaller than the diameter of the lens.

It is customary to express the aperture as a fraction $\frac{f}{x}$ of the focal length: then $\frac{1}{x}$, is the rapidity and the exposure of a sensitized plate must be proportional to x^2.

The use of stops or diaphragms is equivalent to a diminution of the aperture and consequently, of the rapidity. It must be supposed that the effective aperture is equal to the opening in the stop: it depends entirely on the position of the stop with reference to the lens.

The following process due to Steinheil, permits to measure the effective aperture. Focus on a distant object and replace the ground glass by a screen with a hole 1/8 inch in diameter, on the optical axis. Put a light close to and behind the hole; covering the lens with a piece of ground glass, an illuminated circle will be seen which represents the effective opening.

108. CAMERA OF CANADIAN SURVEYS. The camera employ-

ed on the Canadian Surveys (Plate V) is merely a rectangular box of mahogany firmly bound in brass, one face having a hole for the lens and the opposite one being left open to receive the plate holder. The size is the English half plate, 4¾ x 6½ inches. The box is constructed with great care, the faces being perfectly plane and as nearly parallel and perpendicular to each other as they can be made.

It is fixed to the tripod by the screw in the tripod-head: nuts in which that screw fits, are encased in two of the faces of the camera, which may thus be placed with longest dimension either horizontal or upright.

The opposite faces are arranged to receive two levels at right angles: they rest on the face itself and can be changed from one to the other according to the face which is uppermost at the time.

A diaphragm in the middle of the box, cuts off the part of the light admitted through the lens which does not contribute to the formation of the image.

Four fine combs, one eighth of an inch wide, are fixed to the camera immediately in front of the middle of the slides of the plate:they serve to fix the position of the horizon and principal lines. The combs being fixed to the camera and not inside the holder,are about one quarter of an inch in front of the plate:the distance being short and small stops used in the lens,the images of the combs come out sharp on the plate although not in contact with it.

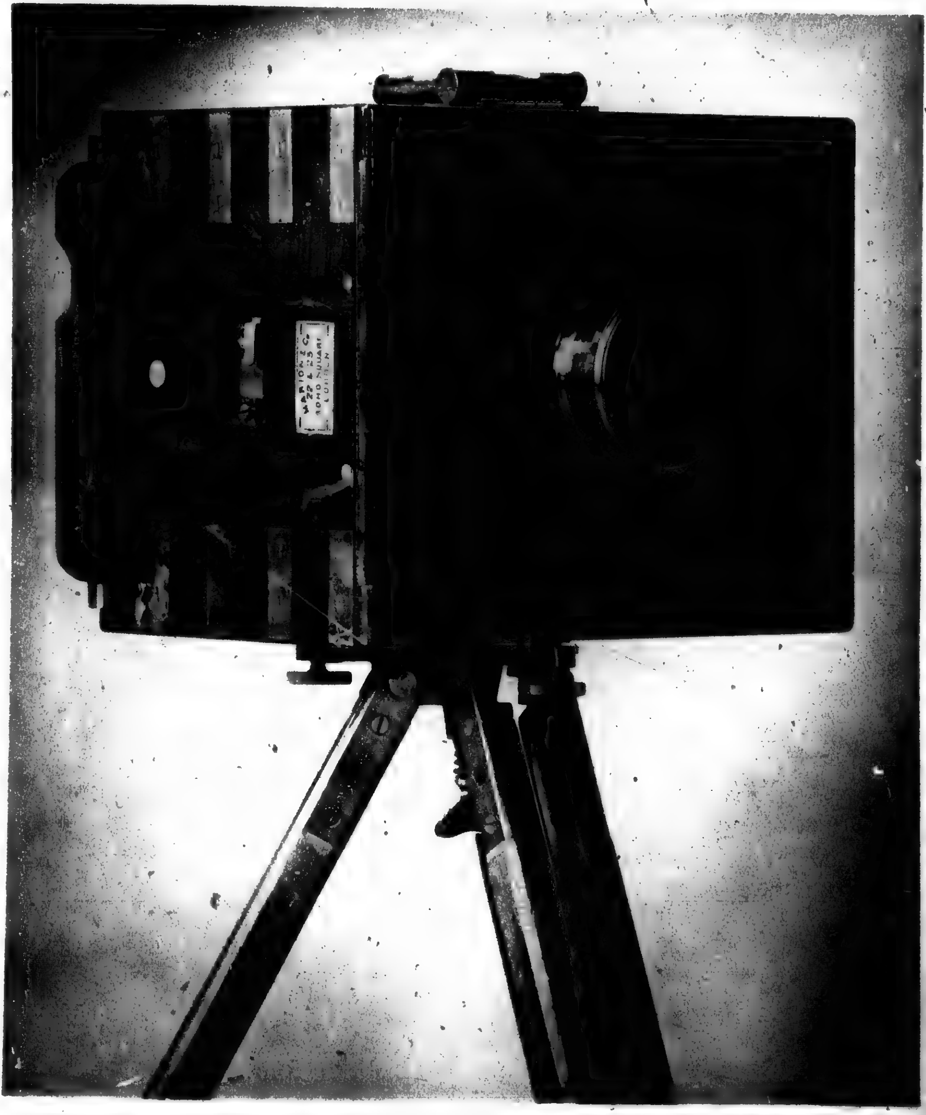
A hood with sun shade can be adapted to the lens, for cutting off the light not required for the image and protecting it from the direct rays of the sun.

The lens is a Dallmeyer wide angle,No.1.A.,of 5½ inch focus:with the half plate employed,it gives an angle of 45° in one direction and 60° in the other.

Three levelling screws,forming part of the head of the tripod,serve to level the camera. Once levelled,it may be turned around,the screw fixing it to the tripod acting as an axis;it remains tolerably level during the revolution.

Two lines marked on each of the faces receiving the levels,indicate the angle embraced by the instrument and enable the surveyor to see what he is taking without the use of the ground glass and black cloth.

PLATE V.



CAMERA OF CANADIAN SURVEYS (HALF SIZE.)

Accompanying the camera are six double holders, containing one dozen of plates. The faces of the holders are numbered from 1 to 12.

109. USE OF CAMERA. Having set up the camera on the tripod and levelled, the surveyor turns it around until the lines on the upper face show him that it is properly directed. He then looks along the lines of the other face to see whether the view will reach high or low enough. If not, he puts the longest dimension of the camera upright, if not already in that position. He now inserts the plate holder and draws the slide, after making sure that the cap of the lens is on. He levels again and takes off the cap. The exposure completed, he puts on the cap introduces the slide and withdraws the holder.

The views are always taken with smallest stop they are perhaps not so brilliant as with a larger stop, but the details are finer and that is of more importance for surveying purposes than artistic appearance.

110. ADJUSTMENTS AND DETERMINATION OF CONSTANTS OF THE CAMERA. There is but one adjustment for the

camera:the sensitized plate must be vertical. The levels being on the upper face,the box must have been made with sufficient precision to ensure the verticality of the plate when the camera's upper face is horizontal.

The verticality can be verified with a plumb line or with a level mounted like a builder's level.

The following process is more precise. Put a piece of plate glass or a good mirror in the holder, in the place occupied by the sensitized plate,and open the back slide. Set up the altazimuth near the back of the camera,fix the telescope to an altitude of 0° and by turning it round,find a point on the same level as the telescope and which can be seen by reflexion in the plate glass of the camera. Then turn the instrument round the vertical axis until the reflected point is seen. If at the intersection of the wires,the plate is vertical.

For let A Fig.144 be the projection of the instrument on the principal plane,M the projection of the point and BC the trace of the reflecting surface of the glass plate;A and M being on the same level, AM is horizontal. When the telescope is turned towards

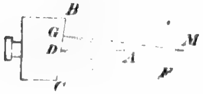


Fig. 144

the glass plate, the horizontal line AD is the projection of the telescope's optical axis: should CB be vertical or in other words should the plate be vertical, a ray of light emitted by the point M and falling in D would be reflected in the direction DA of the optical axis.

Should the top of the plate be inclined backwards, as in figure, MD will be reflected in DF and the point M will cease to be seen on the horizontal thread of the telescope. In order to bring it there, it will be necessary to raise the telescope so as to look in the direction AG: M will show above the horizon. Should the plate incline in the opposite direction the telescope would have to be lowered to set it on the reflected point; M would show below the horizon.

Were the plate found to be inclined, the level perpendicular to it should be adjusted until the change of inclination of the camera brings the plate vertical.

The holders should all be exactly alike, so that the plates may be at the same distance from the lens. Measuring with a scale from the flange of the lens to plates in the holders will show whether this condition has been fulfilled by the maker.

The plate should be in the focal plane of the lens: the test is made in the usual way with a ground glass. If not satisfactory various expedients may be resorted to for moving the lens until proper definition is obtained.

The principal point and the horizon and principal lines occupy the same position on all the photographs taken with the surveying camera: with the distance line, they are the constants of the camera.

The distance line is the equivalent focal length of the lens or, more precisely, the distance from the second nodal point to the sensitized surface of the plate.

The principal point is the foot of the perpendicular let fall from the second nodal point to the surface of the sensitized plate: the horizon and principal lines are the horizontal and vertical lines passing through the principal point.

There are several methods for determining the focal length directly, but unless it be intended to plot from the negatives, the determination should be made from a print similar to those employed for the construction of the plan.

A print is seldom equal in size to the negative: either it contracts in drying or expands in mounting. When the contraction or expansion is uniform in all directions, the figure of the print is similar to the negative and therefore corresponds to the perspective on a plane parallel to the real picture plane, for all perspectives on parallel picture planes are similar. The real picture plane to which the negative corresponds, is the first focal plane of the lens or more accurately, a plane parallel to the sensitized plate at the same distance in front of the first nodal point as the plate is from the second nodal point. A contracted print corresponds to a picture plane nearer to the lens and consequently to a shorter distance line: an expanded print to a more remote picture plane

and longer distance line.

The constants of the perspective required for the construction of the plan, are those which apply to the prints and therefore should be obtained from them; those ascertained directly would be erroneous if applied to the prints.

The determination is made on a view containing some well defined points near the horizon. After taking the view the camera is replaced by the altazimuth and the bearings and altitudes of the points measured.

The distance line, which is the focal length of the lens, is already approximately known; with this value, and the altitudes of the points, the heights of the perspectives α, β, γ, (Fig. 145) above or below the horizon line can be calculated. Laying them in an approximately vertical direction in $\alpha a, \beta b, \gamma c$........, determines an approximate horizon line HH'.

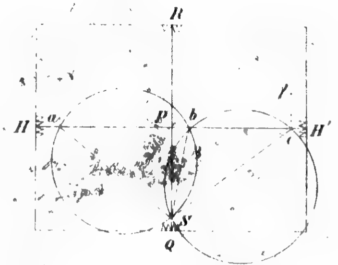


Fig. 145

The horizontal angles between α, β and γ or what is the same, between a, b and c, have been measured from the station. About a and b, describe a circle containing the angle measured between them; also about b and c, a circle containing the angle between the two points. The intersection of the two circles represents the station in the horizon plane, supposed to be revolved on the picture plane around the horizon line as an axis. A perpendicular SP to the horizon line is the distande line and P is the principal point. The perpendicular QPR to the horizon line is the principal line.

The distance line and direction of the vertical which were assumed, can now be rectified, and if necessary, a new construction made which will give more precise values. Other points must also be employed so as to check the result obtained from the first three.

A convenient method for making this determination consists in drawing from a point S, Fig.146 the directions of all the points observed, then on the edge of a sheet of paper made perfectly straight, mark the projections $a, b, c, \ldots\ldots$ (Fig.145) of the perspective on the horizon line. Apply the sheet of paper on the lines drawn from S, (Fig.146) and move the paper until each of the points *a, b, c*...... coincides with the line representing its direction. Mark the position of the extremities of the paper's edge and draw a line through them. This line is the trace of the picture plane on the horizon plane, S being the station.

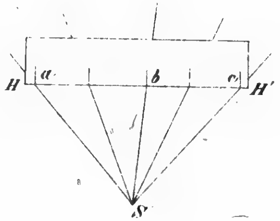


Fig. 146

Instead of constructing the distance line, the polar co-ordinates $bS = \rho$ and $abS = \omega$ of the station S (Fig.147) may be calculated. The triangle abS gives:

$$\rho = m \frac{\text{Sin.}(M+\omega)}{\text{Sin. } M} \quad (1)$$

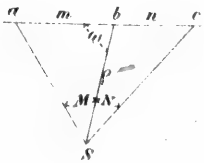


Fig. 147

This is the equation in polar co-ordinates, of the circle described about a and b and containing the angle M, the origin of the co-ordinates being at b.

Similarly, the triangle bcS gives:

$$\rho = n \frac{Sin.(\omega - N)}{Sin.N} \qquad (2)$$

which is the equation of the circle about b and c.

The point of intersection is found by making (1) equal to (2), which after reduction, gives:

$$Tan.\omega = \frac{m + n}{\frac{n}{tan.N} - \frac{m}{tan.M}}$$

The value of ω obtained from this last equation, is employed with (1) or (2), to obtain ρ.

The distance line is equal to

$$\rho\, sin.\,\omega$$

and the principal point is at a distance from b equal to

$$\rho\, cos.\,\omega$$

The horizon and principal lines pass through certain points of the comb marks which are in the middle of the sides of the photograph, having been impressed there by the combs fixed to the camera, a short distance in front of the plate. These points are noted and will serve in future for drawing the two lines without any new determination.

CHAPTER V

PHOTOGRAPHIC OPERATIONS

111. DRY PLATES. The views employed in photographic surveying are those of distant objects:they are the most difficult to obtain by photography. The foreground is easily obtained clear,bright and full of detail,while the distance will come out fogged and indistinct,unless the best plates be employed and skill exercised in the exposure and development. The difficulty is due to the blue haze interposed between the lens and the distant points:blue being a very actinic color,a slight haze unperceptible to the eye, has a most damaging effect on the negative.

It is impossible to give precise directions concerning the kind of plate to be used:photography is progressing so rapidly that in a short time the directions would become obsolete.

A collodion orthochromatic emulsion has,it is said, been discovered recently:if reliable,it would be the best for surveying purposes. The blue mist would not so much affect an orthochromatic emulsion as an ordinary one and the collodion ought to yield fine negatives for enlarging.

Of the plates now on the market,the most suitable are probably the gelatino-bromide orthochromatic dry plates. The ordinary gelatino-bromide plate is most sensitive to blue and violet rays,and little affected by green,yellow and red. A special treatment of the emulsion or plate makes it more uniformly sensitive to all colors;the plate is then said to be "orthochromatic" but the full effect of the treatment requires the use of a yellow or orange transparent screen somewhere in front of the plate. By the process to which it has been submitted the plate has become more sensitive to green,yellow and red rays;

the colored screen, in cutting off a large proportion of the blue and violet rays, increases the effect and finally, the different colors are rendered in nearly their true values.

Among the ordinary plates, the slow ones will give the best results. They should be thickly and evenly coated and allow a very great latitude of exposure. Except the borders covered by the holder, which should be perfectly clear, very little clear glass should appear in the finished negative: a plate in which patches of clear glass are seen yields hard negatives without detail in the shadows. The gradation of the half-tones should be continuous, from nearly opaque black to nearly clear glass. The plates must keep well, and be easy to develop without accident. These qualities are found in several of the slow plates now on the market: with a long exposure and a careful, well restrained development, they will be found to work fairly well.

During the last few years, a great number of films of various kinds have made their appearance. In a mountainous country, nothing better could be desired;

large sizes could be used without increasing the weight to be carried and without the risk of breakage. Unfortunately, there are serious objections. In the first place, no film has yet been produced which may be pronounced equal to a good glass plate and several require after treatments which make the process long, tedious and liable to accidents. Then they are more or less subject to contraction or expansion: as long as it is uniform, it does not affect the accuracy of the views and their fitness for surveying, but it is not certain that this uniformity exists and the least distortion is sufficient to cause their rejection.

The packages of plates received from the manufacturer, are transported in tin boxes hermetically closed: the boxes are always kept locked.

A chinese lantern made of ruby fabric, is used for changing the plates. This is done at night in the surveyor's tent. Should the moon throw too much light, a few blankets thrown on the the tent, make it safe enough. The old plates are taken out and replaced in the original packages; the new plates are marked, introduced in the holders and dusted with a wide camel's hair brush. The holders are kept scrupulously free from dust inside;otherwise the constant motion would bring it on the plate and cause pinholes.

Before introducing a plate in the holder, the numbers of the dozen and of the holder are inscribed in pencil in one of the corners. Figures written with a soft pencil on the film are quite plain after development and have no chemical effect.

112. **EXPOSURE.** The length of exposure is influenced by four causes:

1. The rapidity of the lens.
2. The rapidity of the plate.
3. The strength of the light.
4. The nature of the subject.

The first two causes are constant and their effect may be determined once for all.

The rapidity of the lens is the square of the fraction $\frac{1}{x}$ which expresses the ratio of the aperture to the focal length. The lens and stop used being always the same, the rapidity is constant.

When ordered, it should be specified that the plates are to be of one emulsion: their rapidity will be uniform at first and will change but little afterwards.

Having ascertained by experiment the time of exposure required for a certain subject and with a certain light, it may be assumed that the same exposure will suit any similar subject with a light of equal intensity.

The strength of the light is measured by instruments called photometers. One of the simplest consists of a piece of sensitized paper on the surface of a tinted cardboard, the tint being that acquired by the sensitized paper after some exposure to light. The photometer being exposed to diffused light, the sensitized paper will acquire the same tint as the cardboard after a certain number of seconds: this number indicates the strength of the light and it is assumed that the exposure of a dry plate should be proportional to it. This is not strictly true, the sensitive surfaces not being of the same kind but it is sufficiently accurate in practice.

Decoudun's photometer consists of a screen of graduated opacity interposed between the eye and the ground glass of the camera. By increasing the opacity, a time comes when no light is seen: the opacity then indicates the brightness of the image. The great advantage of this photometer is that it measures at the same time the influence of the last two causes of uncertainty in the time of exposure, the strength of the light and the nature of the subject. It is open to two objections. In the first place the eye is influenced by the light itself and what appears opaque at one time would be translucent under other conditions without any change in the brightness of the image. In the second place the instrument measures the intensity of the luminous rays and indicates a longer exposure for blue than for yellow light while the reverse should be the case. However, the instrument may render good service, particularly with orthochromatic plates.

The nature of the subject varies little in photographic surveying; it is always a distant landscape. The exposure should be timed for the shadows, and those are more or less dense. A deep valley in the shade will require a longer exposure than a wide one.

The color of the surface is an important factor: wooded tracts requiring a very long exposure. Bare rocks are generally of a light color and when at a distance impress the plate quickly:the color being light,its kind does not seem to make much difference in the time of exposure.

In case of doubt,it is safer to err on the side of over exposure:with the proper kind of slow plate, and careful development,a fair result can be obtained from a very much over exposed plate.

After exposure,the plates are forwarded to the Head Office,where they are developed. Accompanying them,are the notes of the surveyor giving the time of exposure for each plate,the strength of the light, whether clear or cloudy weather,the direction of the sun,either in front or behind,the nature of the subject,whether bright or dark and such other information as may help the photographer in the development.

113. DEVELOPMENT. Every photographer has his own favourite developer:it is the best so far as he is concerned. Being used to handle it,he knows how to modify its strength to suit the various conditions

of the plate, and will obtain with it, results superior to those which any other formula would give him. A good plate should work well with any formula and there is no reason why the photographer should not use his own developer.

Negatives intended for enlarging should be clear from stain; although it may be removed by clearing solutions, the additional manipulations are objectionable. For this reason, hydrochinone was adopted as developing agent. The developer is composed as follows;

Sulphite of soda (crystals)	6
Hydrochinone	1.2
Caustic Soda	0.8
Water	92.

To which is added bromide of potassium according to the appearance of the plate.

The negatives, if sufficiently exposed and brought out without forcing, will be composed of pure blacks and whites without any coloration. They are fixed and washed as usual, and are wiped before drying, in order to remove any deposit which may have formed on the film during the various operations to which the plate was submitted. The number of the plate already inscribed in pencil is now written in ink in the margin so as to print with the subject. The plates are kept and stored in envelopes properly numbered and classified.

114. ENLARGING. The print can be made either by contact printing or by enlarging. Contact printing undoubtedly gives the finest prints, while details are sometimes lost in the process of enlargement. On the other hand, measurements are taken direct on an enlargement with a precision which would require the use of a microscope on the contact print; the enlargement also affords room for the construction lines which would soon become confusing on the smaller print.

In so far as the perspective is concerned, an enlargement is the figure which would be obtained on a plane parallel to the picture plane but at a greater distance from the station: thus if a perspective be enlarged twice, it will correspond to a picture plane with a distance line double of the real one.

In the same manner a photograph enlarged twice is the same as one taken direct with a lens having a focal length double that of the lens employed. It is convenient to enlarge in a proportion which will make the distance line an even length:for instance with the negatives taken in the camera described above, the enlargement makes the distance line one foot, which represents 20000 feet on the ground plan (scale $\frac{1}{20000}$).

Positive bromide paper is employed and the enlargement made with a copying lens,in order to secure good detail. The prints are developed in the usual manner,washed and dried and are then classified in scrap books,with pages properly numbered.

The essential condition for a correct enlargement is that the negative be parallel to the easel or board on which the sensitized paper is stretched. The apparatus should be put up so as to fulfil this condition:if not,it may be realized as follows:-

Expose a plate to light and develop it completely dark. With a fine point,draw a square or rectangle on the film and introduce the plate in the enlarging camera. The image received on the easel should be another square or a rectangle similar to the one described on the film; the direction and inclination of the easel must be modified until similarity in the figures is attained.

The easel should be mounted on rails, permitting to displace it without changing its direction.

The apparatus may be set by trial to any desired degree of enlargement, but it is more precise and probably shorter to calculate the positions of the lens and easel.

Let N_1 and N_2 Fig.148 be the nodal points of the lens, P_1 and P_2 two conjugate foci and F_1, F_2 the corresponding focal lengths. Designating the enlargement by α we have:

$$\frac{F_2}{F_1} = \alpha$$

α can be measured directly; for instance with the darkened plate employed

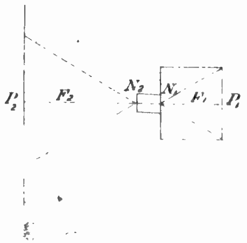


Fig.148

for setting the easel parallel to the negative, by measuring the side of the square or rectangle on the film and on the image.

A trial in another position of the easel will give:

$$\frac{F_2'}{F_1'} = \alpha'$$

The extension of the camera is marked on its bed at each trial: the distance between the marks represents the displacement M of the lens,

$$M = F_1' - F_1$$

The two positions of the easel have also been marked: its displacement, N, is composed of two parts, one due to the change of F_2 and the other to the displacement of the lens: therefore:

$$N = F_2 - F_2' - M$$

We have now four equations with four unknown quantities: deducing their values we find:

$$F_1 = \frac{N + (1 + \alpha')M}{\alpha - \alpha'}$$

$$F_2 = \alpha \frac{N + (1 + \alpha')M}{\alpha - \alpha'}$$

With these two values, the distance of the principal focus is calculated by the formula of conjugate foci

$$\frac{1}{F_1} + \frac{1}{F_2} = \frac{1}{f}$$

To enlarge n times, we must have

$$\frac{f_2}{f_1} = n$$

But we have also:

$$\frac{1}{f_2} + \frac{1}{f_1} = \frac{1}{f}$$

Therefore:

$$f_2 = (n+1)f$$

$$f_1 = \frac{n+1}{n}f$$

The lens has to be moved a distance of

$$f_1 - F_1 = \frac{n+1}{n}f - F_1$$

This is done by measuring this distance from the mark on the bed of the camera indicating the first position of the lens.

The displacement of the easel should be:

$$F_2 - f_2 - f_1 + F_1 = F_1 + F_2 - \frac{(n+1)^2}{n}f$$

Various devices have been proposed to correct the errors due to the distortion of the paper in printing: they generally consist in the impression of a network of squares on the image.

The impression can be made on the negative itself;for instance a ruled glass plate may be introduced in the holder over the sensitized plate,the ruled lines being photographed with the subject.

It would be very inconvenient to carry out such an arrangement;provided a small stop be employed, the same result is attained by means of a net fixed to the camera immediately in front of the holder. Hair has been proposed for the network,but it is too much affected by moisture and is too liable to break to be of much service. A ruled glass plate is probably preferable.

The lines may also be impressed either before or after the exposure on the subject,an additional exposure being given under a ruled plate having clear lines ruled on a non-actinic background,such as collodion mixed with aurine. The lines on the print will show white.

Instead of having the network on the negative, it may be impressed on the print by exposure under a plate similar to the last one.

All these methods do not suppress distortion:

they merely permit to make accurate measurements on the photograph: they would be of little or no use for correcting the errors caused by distortion in most of the construction based on the laws of perspective.

The process employed at the office of the United States Coast and Geodetic Survey for furnishing accurate photographic copies of maps to the engravers, would be preferable.

The print is made a little smaller than the proper size, and while wet, is placed in a frame where it is held by clips. The print is stretched until brought to the proper size and figure, by means of adjusting screws acting on the clips: it is left to dry in that position and pasted on strong cardboard before being removed from the frame.

With prints made on heavy bromide paper, the contraction being tolerably uniform, all these methods of correction which introduce a great complication in the plotting of the plan, may be dispensed with. Moreover, there are other means of counteracting the effect of distortion which will be exposed further on.

The ordinary albumen paper being thin, is more liable to distortion than the heavy bromide paper; for that reason the latter appears preferable even for contact prints.

CHAPTER VI

FIELD WORK

115. TRIANGULATION. The triangulation may be executed at the same time as the topographical survey, but it is preferable to have some of the principal points located in advance by a primary triangulation.

The subject is fully treated in the standard works on surveying; very little requires to be added here. However, there exists some misconception as to the order to be followed in the operations: a few words of explanation may prove useful.

A survey must be considered as consisting of two distinct operations. One has for object the representation of the shape or form of the ground, the other the determination of its absolute dimensions. A perfect plan or triangulation can be made without the measure of any base or length: the plan will exhibit the various features of the ground in their exact proportions, but no absolute dimension can be measured on it until the scale of the plan has been determined. This is done by measuring on the ground one of the dimensions represented on the plan: so the object of the measure of a base is to fix the scale of the survey.

To execute a triangulation, the surveyor is recommended to commence by measuring a base and to make it the side of a triangle, on which he will build others of increasing dimensions. There is a certain logical sequence in the order followed, but in strict theory, the order is immaterial, the triangulation may be executed first and when completed, connected with a base by triangles decreasing in size as they come near the base.

In practice, the case is different: there are several advantages in executing the triangulation before the measure of the base.

The choice of a base must fulfil several conditions: the ground must be tolerably level and free of obstacles, and the direction, length and position of the base must be such as to permit a good connection by triangles of proper shape, with the main triangulation. The surveyor can make a better choice after he has been over the whole ground than on his arrival, when he has seen little of it. Having established the main triangles, he will also know best to connect them with a base. In a mountainous country, the principal summits of the triangulation are fixed by nature and cannot be changed while the position or direction of a base may generally be modified to some extent. Were the base measured first, it might be found not to connect properly with the main triangles.

The secondary triangulation is the work of the topographer, and the construction of signals on the secondary points should be his first act on arriving on the ground.

Should the time at his disposal allow, he will not commence the survey proper until all signals have been established, otherwise he may have to measure angles between points not very well defined. In such a case, the closing error of a triangle is assumed to be due to the want of definition of the points.

Let A, B and C represent the angles of a triangle, whose summits have been occupied in the order given. At A, the surveyor observes the angle between B and C, where there are no signals. He puts up a signal at A and moves to B. In measuring the angle between A and C, he has a signal at A and none at C. Placing a signal at B, he measures the third angle C between two signals.

Call α the closing error of the triangle and ε the probable error of a sight on a point without signal. The probable errors of the angles are:

for A $\varepsilon\sqrt{2}$
" B ε
" C a

The corrections applied to the angles must be proportional to the probable error of each; they are:

for A_1 $\dfrac{\alpha}{1+\sqrt{\frac{1}{2}}}$

for B_1 $\frac{\alpha}{1+\sqrt{2}}$

" C_1 0

The closing error must not exceed a certain limit fixed by the degree of precision of the survey:when the limit is exceeded,the stations must be re-occupied,commencing at the most doubtful one.

The stations of the primary triangulation are the last ones to be occupied when they have been established by a previous survey.

To have a correct idea of the work he is doing, the surveyor must make in the field a rough plot of his triangulation,on which he marks all the stations occupied. It will show him the weak points of the survey and permit him to plan his operations with more assurance.

The object of the secondary triangulation is to fix the camera stations:its summits must be selected for that purpose only. All the topographical details of the plan are drawn from the camera stations.

116. CAMERA STATIONS. A camera station is fixed either by angles taken from the station on the triangulation points or by angles taken from the latter

or by both. It is more easy and more accurate to plot a station by means of angles taken from the triangulation points than by the angles measured at the station, therefore the camera stations should if possible, be occupied before the triangulation summits: there are, however, other considerations which may prevent it.

Camera stations must be chosen in view of the construction of the plan by the method of intersections: other methods are to be employed only when this one fails or when the data collected on the ground are insufficient to furnish a sufficient number of intersections.

A mark or signal of some kind should be left at each station; it does not require to be very elaborate, a pole or a few stones are sufficient. Angles on this signal are measured from the triangulation points, in order to place the station on the plan.

It will seldom happen that the camera is set up precisely at a triangulation point. Generally it will be advisable to move a few feet in one direction or another, for including in the view a certain part of the landscape. Whenever there is any advantage in displacing the camera the surveyor should not hesitate to do so. The distance from the triangulation point measured with a light tape and an angle read on the instrument locate the camera station.

For the same reason, it is not necessary that several views be taken from each station: every view should be taken from the point where it is best for the construction of the plan. The greater number of stations gives very little extra work either in taking the angles for fixing their positions or in plotting them.

In general, views taken from a great altitude and overlooking the country are desirable, but there are numerous exceptions.

Sometimes difficulties may exist in obtaining two views which will furnish intersections over a certain part of the ground. In such a case, the method of vertical intersections may be employed, views being taken from different altitudes. Provided the difference of altitude is large enough and the points to be determined not too far, the precision is the same as with horizontal intersections.

It would be desirable to have the views of the same part of the ground taken at the same time of day. The shadows cast being identical, it is more easy to recognize the different points. It would be well also, to avoid views taken looking towards the sun; they are flat and lack detail. But the surveyor has other considerations to take into account; he will seldom be able to choose his own time to occupy a station or take a photograph. He will often have to take views against the sun or dispense with them altogether. With care in cutting off the sky and giving a long exposure, he may still obtain results remarkably good under the circumstances.

The identification of points, even under different lighting, does not offer any serious difficulties. The number of photographs must be sufficiently large to cover the ground completely: an additional view causes very little extra work, either in printing or plotting and may often save much trouble. The surveyor should not hesitate to take one whenever he finds a place where it may be useful.

Two or three points in each view must be observed with the altazimuth, the altitudes and horizontal angles between them being noted. The altitudes serve to rectify the horizon line on the photograph in case the camera should be slightly out of level and the horizontal angles permit to counteract the distortion of the print by altering the focal length to correspond.

The notes of observations on triangulation points are kept in the usual manner for such work: points of the views are better inscribed on sketches made on the spot. The sketches permit to identify the points with more certainty than a mere designation by a letter or figure.

CHAPTER VII

PLOTTING THE SURVEY

117. SCALE OF PLAN. The minutes of the Canadian Surveys are plotted on a scale of $\frac{1}{20000}$; they are afterwards reduced for publication to $\frac{1}{40000}$. The equidistance is 200 feet.

The convention already adopted in Perspective (§ 54) must be recalled here: the angles measured and the photographs taken must be assumed to have been measured and taken on a model of the ground already reduced to scale.

That the perspectives obtained from any point of such a model will be the same as those taken from the similar point of the ground has already been shown (§ 54);the same rule applies to photographs, in theory at least.

The angles measured are also the same as on the ground,for any triangle ABC,Fig.149,of the ground is represented on the model by a similar triangle *abc*. The altazimuth set in *a* will give between *b* and *c* the same angle as it would between B and C,if set at A.

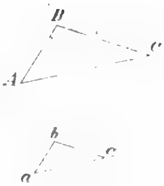


Fig.149

Thus if the plan be required on a scale of $\frac{1}{20000}$ the model will be assumed to have been reduced to that scale and the problem consists in making a plan full size by means of angles and photographs obtained on the model.

No change being made to the camera,the focal length preserves the same value;if one foot,it will cover on the model a distance corresponding to twenty thousand feet on the ground.

The plan and the model being both reduced to the scale of $\frac{1}{20000}$, it is clear that if this scale be used to measure an actual dimension on either, the result is the number expressing the corresponding actual dimension on the ground. If a division of the scale be called a "scale foot", a dimension of the ground is expressed in real feet by the same number which expresses in "scale feet" the corresponding dimension of the model or plan. A distance of a mile contains 5280 real feet on the ground and is represented on the model by 5280 "scale feet"

The focal length of one foot mentioned above would be a focal length of 20,000 "scale feet"

It follows that although the problem consists in representing a model full size, the scale may be employed to measure the actual dimensions, the value of one division being considered as an arbitrary unit.

In other words, a liliputian surveyor must be imagined operating in a liliputian country of which he wants to make a plan full size. His camera is of enormous dimensions, bearing to him the same proportion as a camera several miles long would to an ordinary man.

Of course, all the constructions used in plotting the plan can be demonstrated without such an hypothesis, but the explanations would not be so simple and it would not be so easy to grasp the whole subject.

118. PLOTTING THE TRIANGULATION. The primary triangulation is assumed to have been previously calculated: the primary stations can therefore be plotted at once by their co-ordinates.

The angles of the secondary triangles are now calculated, and the corrections indicated by the closing errors, applied. Some of these triangles have common sides with the primary triangulation: they are calculated first. With the values found for their sides, the adjoining triangles are calculated and so on, until the lengths of all sides have been obtained.

With these values, the differences of latitude and departure from every summit of the secondary triangulation to the nearest primary station are calculated. Unless the primary triangles be very large, the secondary stations can be plotted on the plan by their latitudes and departures without any appreciable error.

The camera stations are next placed by the angles observed upon them from the triangulation points. These angles are plotted with a vernier protractor or by means of a table of chords;either method is accurate enough for the purpose.

As long as a sufficient number of readings have been taken on a camera station from triangulation points,no difficulty is experienced in placing the station:it is not so when only a limited number of readings or none at all are available. There are two cases to consider.

Case I. The camera station has been observed from one or more triangulation points. The camera station M Fig.150,having been observed from the triangulation point A,triangles may be formed with M,A and other triangulation

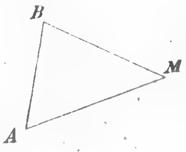


Fig.150

points observed both from A and M, such as B. In the triangle MAB, the angles at M and A have been observed and:

$$B = 180° - (A + M).$$

Similar calculations being made for other triangulation points will give the direction of the station as seen from these points: the plotting is done as if the station had been observed from every such point.

Case II. The camera station has not been observed from any triangulation point. In this case the station must be placed by the angles which have been observed from it. This can be done either by describing through the points observed, circles containing the angles between them, or by the use of a station pointer. The first method requires complicated constructions and is not very accurate and the station pointer can serve only for three points at a time. The following process will be found rapid and accurate when many points have been observed from the station.

On a piece of tracing paper, take a point to represent the camera station and draw the directions of all the points observed. Put the tracing paper upon the plan and try to bring every one of the directions drawn to pass through the corresponding point of the plan. The camera station is then in its place.

From the foregoing, it is clear that the surveyor should endeavour to obtain at least one direction from a triangulation point on every camera station: the plotting is less laborious and the result more accurate.

The use of photographs for placing camera stations must be avoided, the precision is not sufficient.

119. PLOTTING THE TRACES OF THE PICTURE AND PRINCIPAL PLANES. The horizon and principal lines are drawn on the photographs through the proper points of the comb marks (§ 108); the horizon line, however, is checked by the altitudes observed in the same manner as it was placed on the first print when determining the constants (§ 108).

The traces of the principal and picture planes are now drawn on the plan. Every photograph contains at least one and generally several points of which the directions have been observed and marked on the plan. Find the distance Sa, Fig.151, from the station to the projection of such a point α of the photograph on the horizon line; PS is taken on the principal line equal to the focal length and Pa equal to $\alpha a'$.

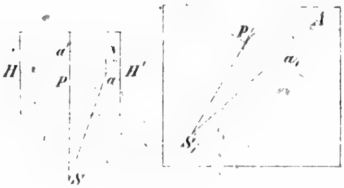


Fig.151

The whole of this construction is made on the "photograph board" which will be mentioned further on.

On the line S_1A of the plan representing the direction of α, take from the station S_1 the distance S_1a_1 equal to Sa: from a as centre with $\alpha a'$ as radius describe an arc of circle and draw S_1p tangent to it: it is the trace of the principal plane. The trace of the picture plane is the perpendicular to S_1p passing through a.

Instead of making the construction on the photograph board, it can be made on the plan. On S_1A take S_1B (Fig.152) equal to the focal length, erect BC perpendicular to S_1A and equal to aa' (Fig.151). Join S_1C and take S_1p equal to the focal length: at p erect a perpendicular to S_1C; it is the trace of the picture plane and S_1C is the trace of the principal plane.

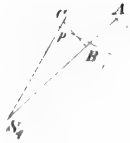


Fig.152

The first method is preferable, because it does not require so many construction lines on the plan.

The trace of the principal plane is marked only where it intersects the picture trace so as not to confuse the plan.

When the directions of several points of the photograph are shown on the plan, either because the directions were observed from the station or inversely because the station was observed from the points, it is better to proceed in the same manner as in § 108 when finding the constants. Mark on the edge of a band of paper the projections of the points' images on the horizon line and try by moving the paper on the plan to place every mark in coincidence with the corresponding direction of the plan; the edge of the paper will then be the trace of the picture plane. The advantage of this method is that a uniform contraction or expansion of the print does not affect the accuracy of the plotting. Even with two directions only shown on the plan, it is advisable to employ the same process, but the two lines not being sufficient to determine the trace of the picture plane, the edge of the paper must be kept perpendicular to the trace of the principal plane drawn as if there were no distortion. This is equivalent to assuming that the contraction or expansion is uniform all over the print.

The effect of distortion may also be corrected to a certain extent by modifying the focal length employed. It has been shown (§ 114) that an enlargement or reduction of the perspective is equivalent to an enlargement or reduction of the distance line. Assuming the contraction or expansion of a print to be uniform, the effect on the perspective is the same as if a shorter or longer focal length had been used.

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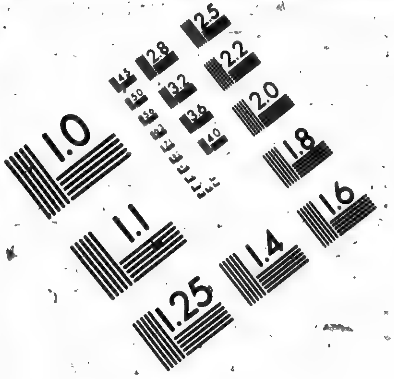


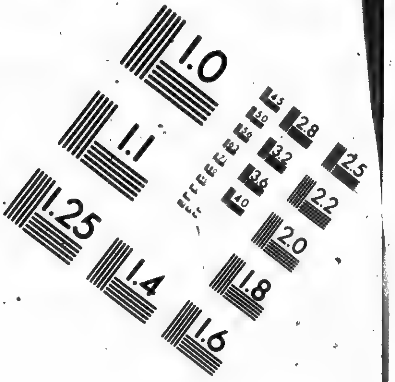
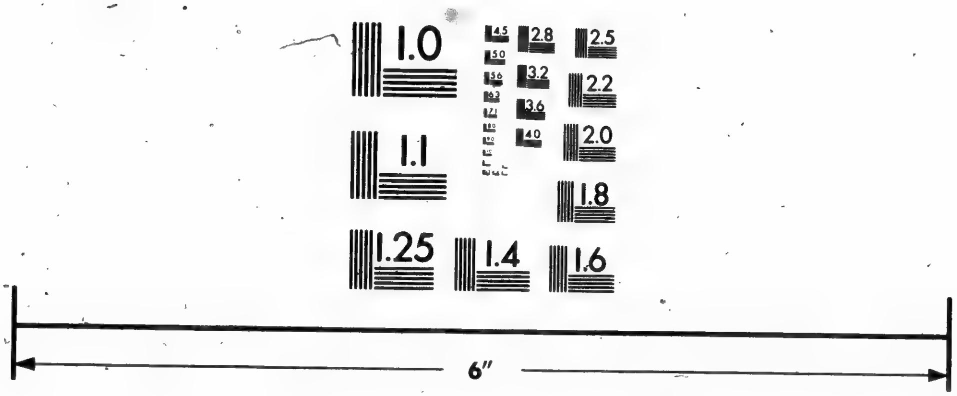
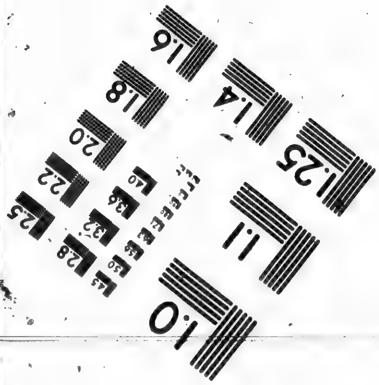
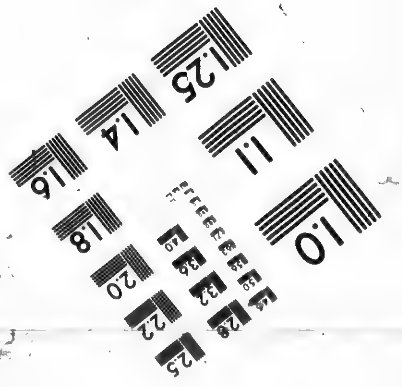


IMAGE EVALUATION TEST TARGET (MT-3)







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The correct length is found as follows.

On the trial print (§ 108) measure the distance between the two comb marks of the horizon line; call it a and let f be the focal length found. Then if the distance of the comb marks be measured on another photograph and found to differ from a, designating by a' this new distance, the enlargement or reduction of the new photograph compared to the trial print is

$$\frac{a'}{a}$$

and we have the proportion:

$$\frac{a'}{a} = \frac{f'}{f}$$

from which the correct focal length f' is deduced.

Should the distortion of the prints be such as to require frequent corrections to the focal length, it might be well to make a scale which applied to the measure of the distance of the comb marks, would give at once the focal length. For instance suppose a and f to be 25,000 and 20,000 feet respectively, on the scale of the plan. Draw a line equal to 25,000 feet and divide it into 20,000 parts: this scale applied to the trial print for measuring the distance of the comb marks, reads 20,000 which is the value of the focal length. Applied to any other photograph it will similarly give the correct focal length.

Wet paper expands more in the direction of its length than in the perpendicular direction. In the case of prints showing an appreciable difference in the rates of expansion or contraction measured on the horizon and principal lines, different focal lengths might be employed, one for the horizon line for plotting the points by intersections, the other corresponding to the principal line, for measuring heights.

In practice, it is found that none of these modes of correction are required when bromide prints on good heavy paper are used, provided they be all treated alike and dried under similar conditions.

120. PLOTTING THE INTERSECTIONS. After drawing on the plan the traces of the principal and picture planes, the draughtsman takes two photographs covering the same ground and marks by a dot and number in red ink the corresponding points of each. The points are chosen on those lines which define best the surface,

such as ridges, ravines, streams, crests, changes of slope etc. He marks on the edge of a band of paper the distance of each point of one of the photographs from the principal line and adjusts the paper on the trace of the picture plane previously drawn on the plan, holding it by paper weights: he repeats the same operation for the other photograph. Inserting a fine needle at each station, he fastens to it a black silk thread connected at the other end by a fine rubber band to a small paper weight. Holding the weight in one hand, he moves the thread until it coincides with one of the marks on the edge of the band of paper corresponding to the station and he deposits the weight on the plan, giving sufficient tension to the rubber to keep the thread taut. Doing the same thing at the other station, the intersection of the two threads indicates the position on the plan of the point of the photographs.

When the bands of paper overlap, as in Fig.153, the portion CD of the picture trace PQ is marked on the band MN which is underneath; the band PQ is placed in proper position and the marks on its edge transferred to the line CD. The band PQ is now placed under MN, the marks on the latter along CD serving the same purpose as those of PQ.

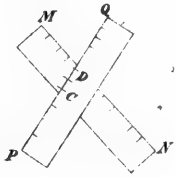

Fig.153

The station may be too close to the edge of the plan for plotting the trace of the picture plane, as for instance A, Fig.154, the picture trace falling in QR, outside of the plan.

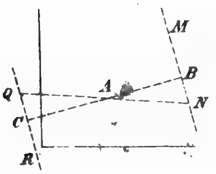

Fig.154

In this case the trace AC of the principal plane is produced to B, a distance equal to the focal length and MN is drawn perpendicular to BC or parallel to QR. The line MN occupies with reference to QR the same position as the focal plane of the camera does to the picture plane of the perspective. The direction of a point of the photograph projected in N on the picture trace, is found by joining NA and producing to the opposite side of A.

The first two intersections should be checked either by a third one or otherwise. They may, for instance be checked by determining the height of the point from the two photographs: unless correctly plotted, the two heights obtained will not agree. This check, however, does not indicate slight errors.

The check may also be a line drawn by means of the perspectograph or perspectometer and on which the point is situated such as the shore of a lake or of a river, but the best check is a third intersection.

The number of every point is inscribed in pencil on the plan.

121. PLOTTING WITH THE PERSPECTOGRAPH. To draw with the perspectograph the plan of a figure which appears on a photograph, the figure must be beyond the picture plane (§ 98) or below the ground line on the photograph. Thus the lake AB (Fig.155), being below the ground line XY of the photograph cannot be drawn with-

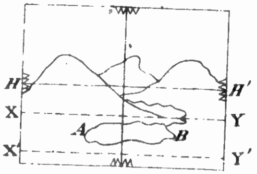

Fig.155

out a change of ground plane, such that the new ground line $X'Y'$ be below the lake AB. It has been explained that this is done by doubling the height of the station until the ground line is brought into correct position (§ 98).

The slide XY of the perspectograph, Fig. 156 is adjusted by the scales drawn in X and Y on the drawing board, to a distance from RT equal to the focal length.

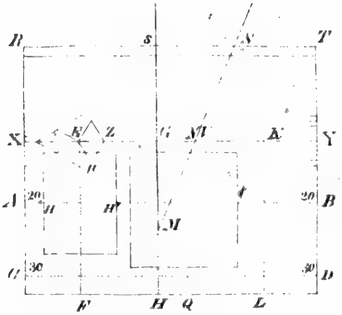

Fig. 156

After adjusting S the pencil is brought over a point, M, of the trace GH of the principal plane at a distance SM from S equal to twice the focal length.

The photograph is pinned under the tracer, the horizon line HH' over the corresponding line AB of the board and the principal line over EF: the iron rod connecting V and Z is then adjusted so as to bring the tracer μ midway between the horizon and ground lines.

The cross section paper is pinned to the board, one of its lines coinciding with the trace of the principal plane GH, and other lines with the front lines AB and CD, drawn at known distances from the foot of the station s.

There will be no difficulty in tracing with the point μ the part of the photograph which on the figure is on the right of the principal line, but it may happen that in moving μ to the left, the obliquity of the arm MS be such as to prevent the free play of the instrument. It should then be reversed, the slide XY being changed end for end, the photograph transferred from EF to KL, the cross section paper moved so as to bring on the trace NQ of the principal plane the line of the paper which was formerly over GH, and the point S placed to the left of s; μ being now between the two slides RT and XY, the tracer has to be changed to the opposite arm.

The perspectograph can be so adjusted that the trace of the principal plane is the same in both positions of the instrument, it being sufficient not to move S, when inverting the arms and slide XY; the cross section paper then does not require to be displaced.

Having obtained the plan of the figure shown on the photograph, the reduction to the proper scale is made at sight on the cross section paper, and transferred to the general plan. The transfer should be checked by points previously established by intersections.

The use of the instrument is possible every time the plane of a figure can be determined, as for instance a lake, a river, a contour line, or the foot of a mountain. Slight differences of level do not affect the result when the height of the station is great.

The instrument could also be used for figures in inclined planes such as a river with a rapid slope, the outline of a stratification plane which has not been distorted, a road or a railway.

122. HEIGHTS. The heights of the points fixed by intersections are found as explained in § 85. The

distance from the point to the horizon line is taken with a pair of compasses on the photograph, one of the points of the compasses is placed on the division A of the sector, (Fig.157), OA being equal to the focal length. The sector is then opened until the other point coincides with the corresponding division B of the other arm. With the same compasses the distance on the plan from the point to the picture line is taken, one leg of the compasses being placed in A on the sector, the other one will come somewhere in C, the compasses are then turned round on C and brought on the division D of the other arm corresponding to C. The line CD is the height of the point above or below the horizon plane, which means the height above or below the station.

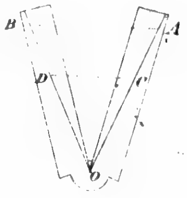

Fig.157

Another method consists in making use of an angular scale as Fig.158. Take SP equal to the focal length; erect the perpendicular PA to SP and divide both into equal parts. Join to S the points of division of PA and through those of SP draw parallels to PA.

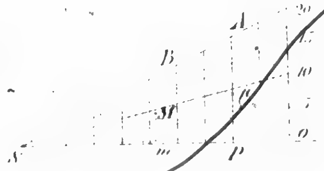

Fig. 158

Now with a pair of compasses, take on the photograph the distance from the point of the perspective to the horizon line: transfer it to Pμ and suppose that it is found to correspond to the line Sμ passing through the point 9 of the graduation of PA.

Take with the same compasses the distance on the plan from the horizontal projection of the point to the picture line and transfer it to P, to the right or left of P according as the point of the plan is beyond or within the picture line. Then take with the compasses the distance on a parallel *m*B to PA, between *m* and the point M where the line *m*B is intersected by Sμ corresponding to 9 of the graduation. This distance *m*M is the height of the point above or below the station.

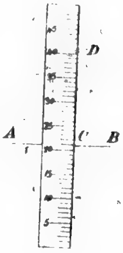

Fig.159

A scale is now pinned somewhere, perpendicularly to a line AB, the division C of the scale corresponding to AB being the height of the station. The compasses are taken off the sector, and one of the legs being set in C, the other leg coincides with a division D of the scale, above or below C, which is the height of the point above the datum plane. This height is entered in pencil on the plan, enclosed in a circle, to distinguish it from the number of the station. It is checked by a second photograph and when the discrepancy between the two heights is within the limits of error admissible, the mean is entered in red ink on the plan and the pencil figures erased.

A difference in the heights obtained from the two photographs indicates that the two points identified do not represent the same point of the ground or that an error has been made either in plotting it or in finding its height.

A third intersection disposes of the first two alternatives and a new measurement of the height shows whether any error has been made.

128. VERTICAL INTERSECTIONS. In the method of horizontal intersections the base line is projected on the horizontal plan:in this method it is projected on a vertical plane. The difference of altitude of the two stations must therefore be considerable.

The principal plane of one of the photographs is taken as vertical plane of projection:the ground plane is the horizontal plane containing one of the stations. In Fig.160,the ground line is the trace of the principal plane of the photograph taken from the station A;the ground plane is the horizontal plane of the station B. On the ground plan, *a* and B are the two stations,CD and EF their picture traces. The station A on the vertical plane is on the

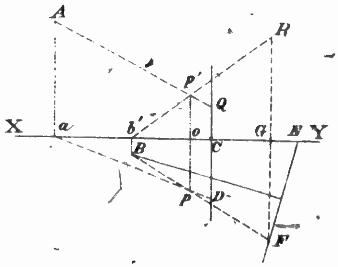

Fig.160

perpendicular aA to XY equal to the height of A above B. A point such as p plotted by the method of horizontal intersections, would not be accurately fixed because the angle of the directions aD and BF is too small.

Project the visual rays from A and B on the vertical plane: the visual ray from A is a line AQ passing through the projection Q of the point's image on the principal line. It is drawn by taking CQ equal to the height on the photograph of the point above the ground line, and joining AQ.

The vertical projection of the visual ray from B is a line b'R passing through the vertical projections of the station b' and of the point's image R, on the second photograph. To find R, let fall FG perpendicular to XY and produce to R, GR being equal to the height on the photograph of the point's image above the horizon line.

The intersection of AQ and b'R is the vertical projection p' of the point. Letting fall the perpendicular $p'o$ to XY and producing, determines the position p of the point on the ground plan.

The construction gives not only the point on the ground plan but also its height *op′*. This process is the best one for plotting a narrow valley between two high walls:it has however the disadvantage of requiring a complicated construction.

124. PHOTOGRAPH BOARD. So many construction lines are employed on the photographs that it is advisable to have a photograph board on which part of the lines are drawn before hand, once for all.

It consists of an ordinary drawing board, covered with strong drawing paper. Two lines at right angles, DD′ and SS′ Fig.161, represent the horizon and principal lines;PD,PD′ PS and PS′ are each equal to the focal length, so that D.D′, S and S′ are the left, right, lower and upper distance points respectively.

The photograph is pinned in the centre of the board, the principal line coinciding with SS′ and the horizon line with DD′. Four scales, forming the sides of a square OTVZ, are drawn in the centre, the side of the square being a little larger than the length of a photograph.

They answer various purposes as, for instance, drawing parallels to the horizon or principal lines by laying a straight edge on the corresponding graduations of the scale or marking the ground line by joining the graduations of the vertical scales representing the height of the station.

At a suitable distance from the distance point D a perpendicular QR is drawn on which are marked by means of a table of tangents, the angles formed with DQ by lines drawn from D. This scale is employed for measuring the altitudes or azimuthal angles of points of the photograph as will be explained later.on (§ 125). From S as a center with SP as radius, an arc of circle PL is described and divided into equal parts. Through the points of division, and between PL and PD′, lines

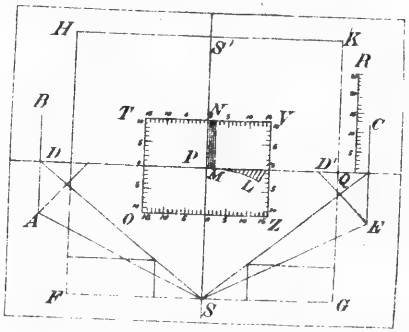

Fig.161

are drawn converging to S. Parallels MN to the principal line are also drawn sufficiently close together. All these lines are used in connection with the scale of degrees and minutes QR.

The studs of the centrolineads are fixed in A, B, C and E; the lines AB and CE, joining their centres and those required for adjusting the centrolineads are drawn and used as explained in § 96.

A square FGKH is constructed on the four distance points.

125. CONSTRUCTION OF THE TRACES OF A FIGURE'S PLANE.

When a figure is in an inclined plane, it is necessary to have the traces of the plane on the principal and picture planes for using a perspective instrument on the photograph.

Two cases are met with in practice: the plane is given by the line of greatest slope or by three points.

Case I. The line of greatest slope may be an inclined road or the middle of a straight valley in which a river flows with a rapid current. On the plan, this line is represented by a line *ab*, Fig. 162, the altitude of a being known.

Pin the photograph to the board and take for ground plane the plane of a: draw the ground line XY.

On the plan draw $a0$ perpendicular to ab and produce it until it intersects the principal line S_1p_1 and picture trace X_1Y_1.

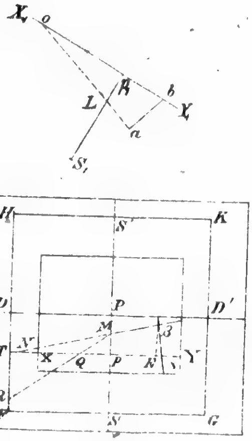

Fig.162

On the photograph take pE equal to p_1b; at E erect a perpendicular to XY and produce it to the intersection β with the perspective of the line of greatest slope. Take pN equal to p_10 and join $N\beta$: it is the trace of the required plane on the picture plane.

Take pQ equal to p_1L and join MQ: it is the trace of the required plane on the principal plane, supposed to be revolved around SS' on the picture plane, the station falling in D. Produce MQ to R: DR is the vertical distance of the station above the plane RMβ. The new horizon and ground lines are now drawn as in § 82.

Case II. Take for ground plane the plane containing one of the points, *a*, for instance (Fig. 163) and draw the ground line XY on the photograph. Join *a* on the plan to the two remaining points and produce to the intersections E and F with the picture trace.

Take pK equal to p_1E and erect KL perpendicular to XY: join the perspectives α and β of the points shown in *a* and *b* on the plan and produce to the intersection with KL. Take p T equal to p_1F, erect TN perpendicular to XY and produce to the intersection N with the line joining the per-

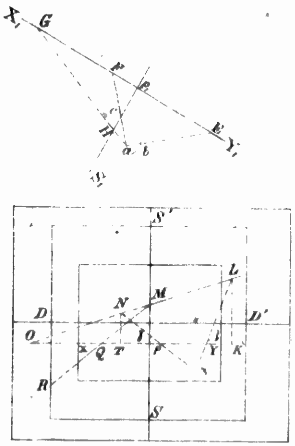

Fig. 163

spectives α and γ.

Join NL: it is the trace of the required plane on the picture plane.

Produce LN to 0 and take p_1G equal to pO; join aG and take pQ equal to p_1H. The line MQ is the trace of the required plane on the principal plane supposed revolved around SS′ on the picture plane, the station being in D. Here also, DR is the vertical height of the station above the plane of the three given points. The new horizon and ground lines are constructed as previously explained.

126. CONTOUR LINES. A sufficient number of heights having been determined, the contour lines are drawn by estimation between the points established. In a rolling country, a limited number of points would permit to draw the contour lines with precision but in a rocky region the inflexions of the surface are so abrupt and frequent that it is utterly impossible to plot enough points to represent the surface accurately. The photographs are of great assistance to the draughtsman; having them under his eye, he is able to modify his curves so as to represent the least in-

equalities of the ground.

Instead of drawing the contour lines at once on the plan, the draughtsman may commence by sketching them on the photograph in the same way as he would on the plan. Every point plotted has been marked on the photograph and the altitudes may be taken from the plan. By adopting this course, he is able to follow very closely the inequalities of the surface. The curves serve to guide the draughtsman in drawing those of the plan or they may be transferred by the perspectograph or the perspectometer.

As long as a sufficient number of points is obtained by intersections, there is no difficulty in drawing the contour lines, but it may happen in a rapid survey, that the points are too few and too far apart for defining the surface. It is then necessary to resort to less accurate methods.

A mountain ridge which appears in $\alpha\beta$ on a photograph (Fig. 164) can be divided by the contour planes, by assuming that it is contained in a vertical plane. The construction, which has been explained in § 62 is carried out as follows:-

On the plan produce the projection *ab* of the ridge, to the intersection F with the picture trace and draw through the station S_1 C parallel to *ab*.

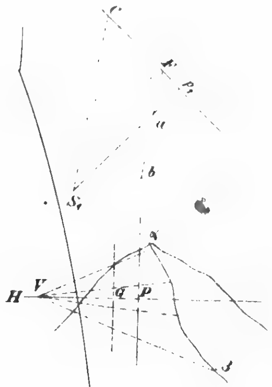

Fig. 164.

Having pinned the photograph to the photograph board, take from the principal point on the horizon line PV equal to p_1C and PG equal to p_1F. At G, place the scale of equidistances perpendicular to the horizon line, the division G corresponding to the height of the station, and join the marks of the scale to the vanishing point V.

Having now the points of intersection of the ridge by the contour planes, their distances from the principal line are marked on the edge of a band of paper and their directions plotted in the usual way. These directions produced to $\alpha\beta$ give the intersections of the contour lines.

When the mountain has rounded forms and no well defined ridge, the visible outline must be assumed to be contained in a vertical plane perpendicular to the direction of the middle of the ridge. The construction is made by drawing, on the photograph board, SV perpendicular to the direction SM of the middle of the outline (Fig.165). On the plan, p_1M_1 is taken equal to PM and from the projection *a* of the summit of the mountain, a perpendicular *ab* is let fall on S_1M_1 which represents the projection of the visible outline: it is produced to the intersection N with the picture trace, PQ is taken equal to p_1N and the scale of equidistance placed at Q perpendicular [illegible] the horiz[illegible] line. The points of division are joined to V, produced to $\alpha\beta$ and the

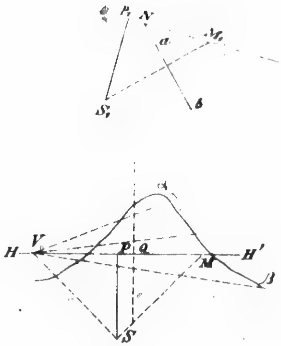

Fig.165

plotting done as in the preceding case, or the directions of the intersections of $\alpha\beta$ by the contour planes may simply be plotted and the contour lines drawn tangent to these directions.

The horizon line contains the perspectives of all the points at the height of the station: it is the perspective of a contour line when the height is that of a contour plane.

Full details on the plotting of contour lines being given in the text books on surveying, it is not necessary to repeat them here. The main point is to understand thoroughly the mode of formation of the surface and its variations under different circumstances: the surveyor should pay particular attention to the subject, making a special study of it. Without this knowledge, the proper representation of the ground would require the plotting of a very large number of points.

127. PHOTOGRAPH PROTRACTOR. The angle between a point of the photograph and the principal and horizon lines, that is the altitude or azimuthal angle, is sometimes wanted.

The azimuthal angle is obtained at once on the photograph board by joining the station S, Fig. 166, to the projection, a, of the point on the horizon line. If required in degrees and minutes, the distance Pa is transferred to the principal line in PG; D is joined to G and produced to the scale of degrees and minutes BC where the graduation K indicates the value of the azimuthal angle.

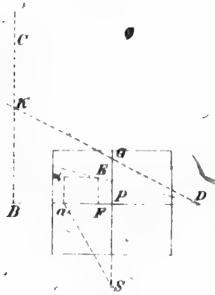

Fig. 166

Were many such angles to be measured, the horizontal scales TV and OZ (Fig. 160) might be divided into degrees and minutes by means of a table of tangents, using as radius the focal length SM. A straight edge placed on a point of the photograph and passing through the corresponding graduations of TV and OZ would at once give the azimuthal angle of the point.

The altitude is the angle S, Fig. 166, of the right angle triangle having for sides Sa and $a\alpha$.

To construct it, take DF equal to $S\alpha$, draw FE parallel and equal to $a\alpha$, join DE and produce to the scale of degrees and minutes BC. This construction is facilitated by the lines previously drawn on the board. With a pair of compasses take the distance from α to the principal line, carry it from P (Fig.160) in the direction PD and from the point so obtained take the distance to the arc ML, measuring it in the direction of the radii marked on the board: this is the distance PF (Fig.166). Then with the same pair of compasses, carry $a\alpha$ to FE which is done by using the parallel lines MN of Fig.160. The construction is now completed as already explained.

A protractor may be constructed to measure these angles: it consists of a plate of transparent material on which are lines parallel to the principal line, containing the points of same azimuth and curves of the points of same altitude.

The azimuthal lines are constructed by plotting the angles in S and drawing parallels to the principal line through the points of intersection with the horizon line.

Denoting by h the altitude of a point α and taking the horizon and principal lines as axes of coordinates, the equation of the curve of altitude h is:

$$y^2 = (x^2 + f^2) \tan^2 h$$

This is an hyperbola of which the principal and horizon lines are the transverse and conjugate axes and the centre is the principal point. One of the branches contains the points above the horizon and the other branch the points of same altitude below the horizon. The asymptotes are lines intersecting at the principal point and making angles equal to h with the horizon line.

This hyperbola is the intersection by the picture plane of the cone of visual rays forming the angle h with the horizon.

The curves of equal altitude may be calculated by the formula of the hyperbola or they may be plotted by points, reversing the construction given above for finding the altitude of α (Fig.166). The complete

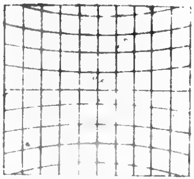

Fig.167

protractor is shown in Fig.167:the angular distance between the lines depends on the degree of precision required.

The instrument may be made,like the perspectometer,by drawing it on paper on a large scale,photographing and making a transparency which is bleached in bichloride of mercury.

128. PRECISION OF THE METHOD OF PHOTOGRAPHIC SURVEYING. The precision of a survey executed by the methods exposed,when all the points are established by intersections,is the same as that of a plan plotted with a very good protractor or made with the plane table. There is however this difference,the number of points plotted by photography is greater than by the other methods.

Points plotted by means of their altitude below the station are far less accurate,their positions being given by the intersection of the visual ray with the ground plane,the angle of intersection being equal to the angle with the horizon plane or to the angle of depression of the point. With the camera employed,embracing 60°,this angle is always less than

30° and even that is seldom obtained in practice, a declivity of 30° being almost a precipice. Therefore the intersection is always a poor one and the uncertainty becomes considerable with points near the horizon.

With perspective instruments, doing mechanically the same construction, the results are still less precise, being affected by the instrumental errors.

On the other hand, it must not be forgotten that when these methods are employed, the ordinary topographer would fall back on sketching; the results furnished by photography therefore are infinitely more precise.

CHAPTER VIII

PHOTOGRAPHS ON INCLINED PLATES

129. Hitherto it has been assumed that the photographs used for the survey were taken on plates perfectly vertical. There are several cases in which this condition cannot be fulfilled: the camera may be an ordinary one, without any means of adjusting the plate, or the photographs may have been taken merely as illustrations, their employment for the construction of the plan being decided afterwards.

There are two classes of surveys in which the plates are always inclined. The first are secret surveys, the views being taken with a camera concealed about the person or otherwise. The scope of these surveys is very limited;the photographs,being instantaneous,lack detail in the distance and unless objects present great contrasts of light and shade, their images are blurred,and confuse as soon as the distance attains a few hundred yards. Improvements in dry plates will no doubt remove this difficulty to some extent,but it will never disappear completely. Another cause of trouble is the small size of the camera and plates:the views,being instantaneous, stand very little enlargement and the measurements are in consequence not very accurate.

The other class of surveys comprises those made from balloons. It is very doubtful whether the method will ever be found practical and prove of more than theoretical interest. It requires the consideration of an entirely new system of survey by means of photographs taken on plates placed horizontally or nearly so.

130. PLOTTING THE DIRECTIONS OF POINTS OF THE PHOTOGRAPHS. When the photographic plate is not vertical,the corresponding picture plane of the

perspective, which is parallel to the plate is pierced by the vertical of the station. This trace is the vanishing point of all the vertical lines, which having ceased to be front lines, are no longer represented by parallels to themselves.

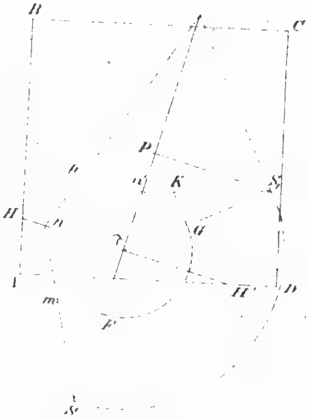

Fig. 168

Let ABCD Fig. 168, be a photograph on an inclined plate, P being the principal point and HH' the horizon line. The perpendicular $V\pi$ drawn through the principal point to the horizon line, is the principal line.

Revolve the principal plane on the picture plane around the principal line as an axis: the station falls in S_1 on a perpendicular PS_1 to VP, PS_1 being equal to the focal length.

Join $S_1\pi$ and S_1V; the first line is the revolved horizontal line from the station to the picture plane;

S_1V is the revolved vertical of the station and V the vanishing point of vertical lines.

Revolve now the horizon plane on the picture plane around the horizon line. The station comes in S, on the principal line produced, at a distance πS equal to πS_1.

To find the horizontal direction of a point μ of the photograph, draw the perspective of its vertical line by joining it to V. The intersection n with the horizon line is the perspective of the trace in the horizon plane of the vertical of the point and Sn is its direction.

Comparing this construction with the one for vertical plates, we see that the same methods may be employed provided π be used as principal point, πS_1 as focal length and that every point of the photograph be first projected on the horizon line by joining it to V, before measuring its distance from the principal line. The points such as n can be marked on a band of paper and used as in the case of vertical plates.

With a plate nearly vertical, V is at a great distance from P, and the perspectives of the vertical lines have to be drawn with the centrolinead.

131. DETERMINATION OF HEIGHTS. Let m Fig. 168 be on the ground plan the point seen at μ on the photograph. Project on the principal plane the triangle formed by the visual ray, its projection on the horizon and the line $n\mu$. On the revolved principal plane, the projection of the visual ray is $S_1 n'$, $\mu n'$ being perpendicular to $V\pi$; the projection of m is F which is revolved to G and the perpendicular GK to $S_1\pi$ is the projection of the vertical of the point or its height above the horizon plane.

Various devices may be imagined for constructing expeditiously the heights of a number of points.

132. DETERMINATION OF THE HORIZON LINE AND VANISHING POINT OF VERTICALS. In order to make use of a photograph for plotting the plan, the horizon and principal lines and the vanishing point of verticals must be marked on the photograph.

It is assumed that the camera is available either before or after the survey, for experimenting upon and that the focal length and principal point may be determined by the usual methods, with the plate vertical. If the zenith distances of several points of the photograph have been observed with a surveying instrument, the determination of the horizon line presents no difficulty. Assume a vanishing point of vertical lines V, Fig. 169, and join it to a point μ of the photograph of which the zenith distance is known. Through the principal point, draw PE perpendicular to $V\mu$ and PS perpendicular to PE, and equal to the focal length. Erect EG perpendicular to ES, take EF equal to $E\mu$ join SF and make the angle FSG equal to the altitude of μ; FG is the distance measured on $V\mu$ from μ to the horizon line.

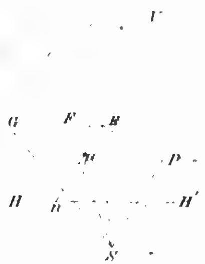

Fig. 169

Taking μn equal to FG determines one point n of horizon line. A similar construction repeated on another point of the photograph will furnish a second point of the horizon line.

This first result will probably be inaccurate because the position of the vanishing point V is only approximate. A new vanishing point must therefore be fixed by means of the horizon line just obtained, and the construction explained above is repeated. The second horizon line found will likely be sufficiently precise; if not the construction must be made a third time.

In secret surveys, measured angles are seldom available, but it is easy to devise an attachment of the same kind as some hand levels, which will mark the horizon line on the plate when the photograph is taken.

Failing this, the horizon line must be furnished by the subject. When the view includes buildings, the vanishing point of verticals is given at once by producing to their intersection the vertical lines of the buildings. This point V, Fig. 170 is joined to the principal point, P, and PS is made perpendicular to VP and equal to the

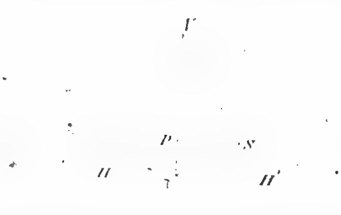

Fig. 170

focal length. Drawing $S\pi$ perpendicular to SV. the perpendicular HH' to $V\pi$ is the horizon line.

Horizontal lines vanish on the horizon line, therefore if the horizontal lines of two faces of a building be produced to their intersection, the line joining the two vanishing points is the horizon line.

If two intersecting horizontal lines appear on the photograph as a straight line, the latter is the horizon line.

Although angles cannot be measured, it may be possible to ascertain the points of the view which are at the same altitude as the observer: these, joined together give the horizon line.

133. TRANSFERRING THE PERSPECTIVE TO A VERTICAL PLANE. Instead of using exact copies of the negatives for plotting the plan, the copies or enlargements can be made in such a way that the perspective is restored to a vertical plane.

Have a copying or enlarging camera OCD, Fig. 171 movable on a horizontal axis passing through the first nodal point O and parallel to the negative.

Take an experimental negative with the field camera, the plate being vertical; draw on it the horizon and principal lines, place it in the holder of the copying camera and mark the points of the holder corresponding to the horizon and principal lines. After inserting the holder, the camera is moved until the plate CD is vertical and fixed in that position. The screen AB is now adjusted at the proper distance, parallel to the plate and the projected images of the horizon and principal lines are marked on it in such a manner that the marks will appear on the prints.

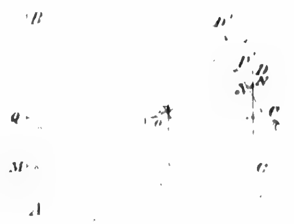

Fig. 171

For copying a negative taken in an inclined position, the horizon and principal lines are drawn on it, also a parallel to the horizon through the principal point. The negative is placed in the holder with the principal line on the proper marks and the horizontal line of the principal point on the marks corresponding to the horizon line of the experimental plate. The camera is moved up or down until the image of the negative's horizon line π coincides with the horizon line Q previously marked on the screen:in this position the perspective is the same as it would have been on a vertical picture plane. For,the inclination of the camera is the same as when the negative was taken:any point N' of the latter would have photographed in N on a vertical plate and given the same image M on the screen.

With a lens of sufficiently long focus and photographs taken nearly vertical,as is generally the case,the displacement of the camera will be too small to affect the definition on the screen.

The holder must be provided with means of adjusting the negative;the principal point must always occupy the same position,the plate pivoting around it.

The horizon and principal lines are indicated on the print by the marks fixed to the screen:the principal point has been displaced in copying and is now on the horizon line.

The change of picture plane can also be effected with the perspectograph, but the use of the instrument is not to be recommended when the change can be made so simply by the photographic process.

134. PHOTOGRAPHS ON HORIZONTAL PLATES. Photographs on horizontal plates might be obtained by an arrangement similar to the one described in § 99, with a pinhole stop in the lens: they are also taken from a balloon with an ordinary camera, but the plates are only approximately horizontal.

The picture and ground planes being parallel, the figures of one are similar to those of the other: thus the photograph $\alpha\beta$ Fig. 172 of a lake AB is also its plan and only requires to be reduced to the proper scale. The reduction is given by the proportion between the distances Ss and SP from the station to the ground and picture planes. When the height of the station and the focal

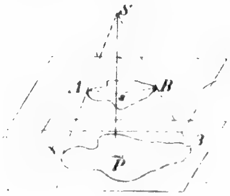

Fig. 172

length are equal, the photograph is a full size plan.

To plot the directions of the various points, the principal point P of the photograph is placed on the foot of the station *s*, and a line of known direction, such as Pα, on the corresponding line of the plan *s*A. To find the direction of any other point B, its perspective β is joined to the principal point P: this line coincides with *s* B on the plan.

Fig. 173

The height of a point is found by taking SP, Fig. 173, equal to the focal length and S*s* equal to the height of the station, drawing Pα and *s*α perpendicular to SP, Pα being equal to the distance of the point's perspective from the principal point and *s*α equal to the distance on the plan from the station to the point. Join Sα; the parallel *a*A to SP is the height of the point above the ground plane.

A photograph taken from a balloon cannot be perfectly horizontal; to make use of it for plotting the

plan, the trace s, Fig. 174, of the vertical of the station on the picture plane must be known.

The directions of the principal line sP and of the perpendicular to it, AB, are the same on the plan and on the photograph; they are different for all other lines.

Fig. 174

To find the direction on the ground plan of a point μ of the perspective, draw PS perpendicular to the principal line and equal to the focal length, join Ss and take SC equal to the distance from μ to AB. Draw μA and CD parallel to Ps and take $A\mu'$ equal to SD; $s\mu'$ is the direction of the point on the ground plan, for $A\mu$ forms with its horizontal projection a right angle triangle in which the angle A is the inclination of the plate to the horizon, which triangle is constructed in SCD, therefore $A\mu'$, which is made equal to SD is the horizontal projection of $A\mu$, μ' is the trace, on the ground plane, of the vertical of μ and the vertical plane passing through s and the point μ of the photograph must cut the ground plane along $s\mu'$.

A much better way to employ these photographs would be to restore them to a horizontal plane in printing, by the process of § 131, using Ps and AB in the same manner as the principal and horizon lines of the vertical photograph.

The great difficulty in balloon surveying will be to determine the trace of the vertical of the station on the picture plane, or the foot of the station on the ground plan. The oscillations of the balloon prevent the use of any kind of level inside of the camera and instrumental measurements of angles are open to the same objection. The angles might, however be measured by two observers located on the ground.

In a view containing vertical lines, their vanishing point gives at once the trace of the vertical of the station; for a photograph taken a short distance above buildings, this mode of determining the trace would be very good.

Balloon surveying would only be adapted to military purposes, although the advocates of the process are confident that it will eventually take the place of all other surveying methods.

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